ON NON-VANISHING OF CERTAIN L-FUNCTIONS

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ABSTRACT

On Non-Vanishing of Certain L-Functions
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This thesis presents the following:

(i) A detailed exposition of Rankin's classical work on the convolution of two modular L-functions is given;

(ii) Let $\bar{\mathcal{S}}$ be the class of Dirichlet series with Euler product on $Re(s) > 1$ that can be continued analytically to $Re(s) = 1$ with a possible pole at $s = 1$. For $F, G \in \bar{\mathcal{S}}$, let $F \otimes G$ be the Euler product convolution of F and G. Assuming the existence of analytic continuation for certain Dirichlet series and some other conditions, it is proved that $F \otimes G$ is non-vanishing on the line $Re(s) = 1$;

(iii) Let $\mathcal{F}_N$ be the set of newforms of weight 2 and level N. For $f \in \mathcal{F}_N$, let $L(\text{sym}^2 f, s)$ be the associated symmetric square L-function. Let $s_0 = \sigma_0 + it_0$ with $1 - \frac{1}{46} < \sigma_0 < 1$. It is proved that

$$C_{s_0,\epsilon} N^{1-\epsilon} \leq \#\{f \in \mathcal{F}_N;\ L(\text{sym}^2 f, s_0) \neq 0\}$$

for prime N large enough. Here $\epsilon > 0$ and $C_{s_0,\epsilon}$ is a constant depending only on s_0 and ϵ.

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NOTATIONS

$f(x) \ll g(x)$ or $f(x) = O(g(x))$ if there exists a constant C such that $|f(x)| \leq Cg(x)$

$f(x) \sim g(x)$ as $x \to +\infty$ if $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1$

$\zeta(s)$: the Riemann zeta-function; page 2.

χ : a Dirichlet character; page 2.

g.c.d.(a, b) : the greatest common divisor of a and b; page 2.

χ_0 : the trivial character; page 3.

$L_\chi(s)$: the Dirichlet L-function associated to a character χ; page 3.

$\pi(x)$: number of primes $\leq x$; page 3.

$\Gamma(s)$: the gamma-function; page 4.

$\theta(x)$: the classical theta-function; page 4.

$\Lambda(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$; page 4.

$\mathbb{H}$: the upper half-plane; page 6.

$GL_2^+(\mathbb{R})$: the multiplicative group of 2×2 matrices with real entries and positive determinant; page 6.

$\mathbb{H}^* := \mathbb{H} \bigcup \mathbb{Q} \bigcup \{\infty\}$; page 6.

"$|_k$" : the stroke operator; page 6.

$\Gamma = SL_2(\mathbb{Z})$: the multiplicative group of 2×2 matrices with integer entries and determinant 1; page 6.

$q_M := e^{\frac{2\pi i z}{M}}$; page 7.

$\hat{a}_f(n)$: the n-th Fourier coefficient of a cusp form f; page 7.

$\Gamma_0(N)$: the subgroup of $\Gamma = SL_2(\mathbb{Z})$ consists of matrices $(a_{ij})_{2\times 2}$ in which a_{21} is divisible by N; page 7.

$M_k(N)$: the space of modular forms of weight k and level N; page 7.

$S_k(N)$: the space of cusp forms of weight k and level N; page 7.

$\langle f, g\rangle$: the Petersson inner product of f and g in $S_k(N)$; page 8.

$D_0(N)$: a fundamental domain for $\Gamma_0(N)$; page 8.

$e(z) := e^{2\pi i z}$; page 8.

$L_f(s)$: the L-function associated to a cusp form f; page 8.

$a_f(n)$: the n-th normalized Fourier coefficient of a cusp form f; page 8.

W_N : the Atkin-Lehner involution; page 8.
$S_k^+(N)$: the $(-1)^{\frac{k}{2}}$-eigenspace of W_N in $S_k(N)$; page 9.
$S_k^-(N)$: the $(-1)^{\frac{k}{2}+1}$-eigenspace of W_N in $S_k(N)$; page 9.
$\Lambda_f(s) := \left(\frac{2\pi}{\sqrt{N}}\right)^{-s}\Gamma(s)L_f(s)$; page 9.
$L_{f,\chi}(s)$: the twisted L-function associated to f and χ; page 9.
$a \mid b$: a divides b; page 10.
$a \nmid b$: a does not divide b; page 10.
$T_p\ (p \nmid N), U_q\ (q \mid N)$: the Hecke operators; page 10.
$\mathbf{d}(n)$: the number of positive divisors of n; page 11.
ϵ_p and $\bar{\epsilon}_p$ $(p \nmid N)$: roots of the quadratic equation $1 - a_f(p)x + x^2 = 0$; page 11.
$S_k^{\text{old}}(N)$: the space of oldforms of weight k and level N; page 12.
$S_k^{\text{new}}(N)$: the space of newforms of weight k and level N; page 12.
$\mathcal{F}_N$: the set of normalized newforms of weight k and level N; page 12.
$\Delta(z)$: the discriminant function; page 13.
$E(z,s)$: the Epstein zeta-function; page 17.
$\Theta(\omega) = \Theta(z,\omega)$: the theta-function; page 18.
$\hat{f}$: the Fourier transform of f; page 18.
$|A|$: the determinant of a matrix A; page 19.
$\text{diag}[a_1,...,a_n]$: the diagonal matrix with entries $a_1,...,a_n$ on its main diagonal; page 19.
$\mathcal{J}$: the Jacobian matrix; page 19.
$\xi(z,s) := \left(\frac{\pi}{y}\right)^{-s}\Gamma(s)E(z,s)$; page 21.
$\delta(f,g) := y^{k-2}f(z)\overline{g(z)}$; page 23.
$L(f \times g, s)$: the Rankin-Selberg convolution; pages 12 and 24.
$L(f \otimes g, s)$: the modified Rankin-Selberg convolution; page 24.
$\zeta_N(s)$: the Riemann zeta-function with the Euler p-factors corresponding to $p \mid N$ removed; page 24.
$F_N(z,s) := 1 + \sum_{m=1}^{\infty} \sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,mN)=1}}^{\infty} \frac{1}{|mNz+n|^{2s}}$; page 26.
Γ_∞ : the stabilizer of ∞ under the action of Γ on upper half-plane; page 26.
$\mathcal{T}$: a set of representatives for right cosets of Γ_∞ in $\Gamma_0(N)$; page 27.
$\mu(n)$: the Möbius function; page 28.

$\Phi(s) := \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)L(f \otimes g, s)$

$\quad = \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)\zeta_N(2s)L(f \times g, s)$; page 30.

$L(\mathrm{sym}^2 f, s)$: the symmetric square L-function associated to a normalized eigenform f; page 35.

$L_\infty(\mathrm{sym}^2, s) := \pi^{-\frac{3}{2}s}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)$; page 36.

$\Lambda(\mathrm{sym}^2 f, s) := N^s L_\infty(\mathrm{sym}^2, s)L(\mathrm{sym}^2 f, s)$; page 36.

$\mathcal{S}$: the Selberg class; page 38.

$a_F(n)$: the n-th coefficient of the Dirichlet series $F(s)$; page 38.

$b_F(p^k)$: the p^k-th coefficient in the p-factor of the Euler product of $F(s)$; page 38.

$\bar{\mathcal{S}}$: the class of Dirichlet series defined in Section 3.2; page 38.

$\bar{F}(s) := \overline{F(\bar{s})}$; page 39.

$(F \otimes G)(s)$: the Euler product convolution of F and G; page 39.

$L(F \otimes G, s)$: the L-convolution of F and G; page 54.

$L(f, s)$: the formal L-series attached to an arithmetic function $f(n)$; page 55.

$(f * g)(n)$: the Dirichlet convolution of two arithmetic functions $f(n)$ and $g(n)$; page 55.

$\int_{(c)} g(s)ds$: the contour integral; page 61.

$\omega_f := \frac{(4\pi)^{k-1}}{(k-2)!}\langle f, f\rangle$; page 68.

δ_{mn} : the Kronecker symbol; page 69.

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Chapter 1

Introduction and Statement of Results

An L-function is, informally, a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ that extends analytically to the whole complex plane and satisfies a certain symmetric relation (i.e., functional equation). The study of these complex functions is intimately related to some important problems in arithmetic and geometry. In particular, the investigation of the zeros of L-functions has played a significant role in the development of modern number theory.

In the first section of this chapter we give a brief historical account of the non-vanishing of L-functions and the close connection of these functions with classical distribution problems in number theory. Through this we try to provide enough motivation for the reader to follow the subject. Since our focus in this thesis is mostly on the non-vanishing of L-functions associated to modular forms, a summary of the basic definitions and results in the subject of modular forms is given in Section 1.2. Section 1.3 is devoted to a summary of the main results of this thesis.

1.1 Riemann Zeta-Function and Dirichlet L-Functions

The proof of infiniteness of primes goes back to Euclid's time. However, it was Euler who, for the first time, studied this discretely natured problem by continuous tools. To do this he introduced the zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for the real variable $s > 1$.

Euler observed that this function has a product representation in the form of

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$$

for $s > 1$. Such a product is called an *Euler product.* By taking the logarithm of both sides of this product, applying the expansion

$$-\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k},$$

and using the divergence of the harmonic series, Euler showed that

$$\lim_{s \to 1^+} \left(\sum_p \frac{1}{p^s} + \sum_p \sum_{k \geq 2} \frac{1}{kp^{ks}} \right) = +\infty.$$

He argued that since the second sum is convergent for $s = 1$, then

$$\lim_{s \to 1^+} \left(\sum_{p \text{ prime}} \frac{1}{p^s} \right) = +\infty.$$

This proves the existence of an infinite number of primes, and proves further that the series $\sum_p p^{-1}$ diverges.

After Euler it was Dirichlet who made major progress in this subject. His investigation in the problem of distribution of primes in arithmetic progressions led him to the notion of a character χ, which now bears his name.

A *Dirichlet character* χ *(*mod b*)*, is a complex function defined on integers such that for any m and n,

(i) $\chi(mn) = \chi(m)\chi(n)$;

(ii) $\chi(n + b) = \chi(n)$;

(iii) $\chi(n) \neq 0$ if and only if g.c.d.$(n,\ b) = 1$.[1]

[1]g.c.d.(a, b) stands for the greatest common divisor of a and b.

If $\chi(n) = 1$ for any n with g.c.d.$(n,\ b) = 1$, then χ is called the *trivial character* (mod b) and it is denoted by χ_0.

Dirichlet realized that in order to study the distribution of primes in an arithmetic progression with common ratio b, one needs to consider the following infinite series

$$L_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $s > 1$. This is known as a *Dirichlet L-function.* It turned out that for any χ, $L_\chi(s)$ has the following Euler product

$$L_\chi(s) = \prod_p (1 - \chi(p)p^{-s})^{-1},$$

and for any nontrivial character χ, $L_\chi(s)$ has a finite value at $s = 1$. By using these facts about $L_\chi(s)$, Dirichlet proved that there are infinitely many primes in an arithmetic progression $a,\ a+b,\ a+2b, \cdots$, when g.c.d.$(a,\ b) = 1$. The central idea in the proof of Dirichlet's Theorem is that for any character $\chi \neq \chi_0$,

$$L_\chi(1) \neq 0.$$

It is accurate to say that this is the first non-vanishing theorem of this type in the history of number theory.

In 1859, twenty years after Dirichlet's work, Riemann published a short and very inspiring article about the problem of the distribution of prime numbers and its connection with the zeta-function. More than half a century ago, Legendre and Gauss independently conjectured that the number of primes in the interval $[1,\ x]$, for large x, behaves like $\dfrac{x}{\log x}$. More precisely, if we define

$$\pi(x) = \sum_{\substack{p \leq x \\ p \text{ prime}}} 1,$$

then

$$\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1.$$

This is the celebrated *Prime Number Theorem.* Note that log here stands for natural logarithm.

In his paper, unlike Euler and Dirichlet, Riemann considered $\zeta(s)$ as a complex variable function. Today, this function in called the *Riemann zeta-function.* In this seminal paper, known as Riemann's Memoir, he outlined a project to prove the Prime Number Theorem. Riemann first started with the definition of the gamma-function at point $\frac{s}{2}$,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t} t^{\frac{s}{2}-1} dt.$$

Using the change of variable $t \mapsto \pi n^2 x$, multiplying both sides by $\pi^{-\frac{s}{2}} n^{-s}$ and taking sum over n's, for $Re(s) > 1$, he arrived at

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty \left(x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} \right) dx. \tag{1.1}$$

Then he utilized the classical *theta-function*

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}.$$

It was proved by Jacobi that $\theta(x)$ has the following transformation property,

$$\theta\left(\frac{1}{x}\right) = x^{\frac{1}{2}}\ \theta(x)$$

valid for $x > 0$.

Riemann applied this property of $\theta(x)$ in (1.1) to derive the following integral representation for the zeta-function,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_1^\infty \omega(x) \left(x^{\frac{s-2}{2}} + x^{-\frac{s+1}{2}} \right) dx + \frac{1}{s(s-1)}$$

where $\omega(x) = \frac{\theta(x)-1}{2}$. From this he concluded that the zeta-function extends analytically to the whole complex plane, except for a simple pole at point $s = 1$. He also proved that the zeta-function satisfies the following functional equation

$$\Lambda(s) = \Lambda(1-s) \tag{1.2}$$

where

$$\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

In the rest of his paper, Riemann made six conjectures, some about concrete properties of $\zeta(s)$, and others relating the zeta-function to the distribution of prime numbers. By the late nineteenth century, four of these conjectures were proved, and as a result, the proof of the Prime Number Theorem was established.

The Prime Number Theorem can be considered as a prototype example that describes the connection between the distribution problems arising in number theory and the non-vanishing of various L-functions of number theory. One of the main steps in the proof of the Prime Number Theorem by Hadamard and de la Vallée Poussin in 1896 was the fact that $\zeta(s)$ has no zeros on the line $Re(s) = 1$. It is not difficult to prove that the Prime Number Theorem implies the non-vanishing of $\zeta(s)$ on the line $Re(s) = 1$. The question arises whether the Prime Number Theorem can be proved using just the fact that $\zeta(s)$ has no zeros on the line $Re(s) = 1$. This was answered affirmatively around 1930 by the work of Wiener using Fourier analysis. So one can say that the Prime Number Theorem is equivalent to the non-vanishing of the Riemann zeta-function on the line $Re(s) = 1$.

The Riemann conjecture about the place of zeros of the Riemann zeta-function is in fact much stronger. This deep conjecture, known as the *Riemann Hypothesis*, asserts that apart from the trivial simple zeros at points $s = 0, -2, -4, \cdots$, all the other zeros of $\zeta(s)$ lie on the vertical line $Re(s) = \frac{1}{2}$. The proof of this conjecture would result in the Prime Number Theorem with the best possible error term. More precisely, assuming the Riemann Hypothesis, one can show that there is a constant C, such that for large x,

$$|\pi(x) - \frac{x}{\log x}| \leq C x^{\frac{1}{2}} \log^2 x.$$

The functional equation for Dirichlet L-functions was first given by Hurwitz in 1882, although he confined himself to real characters. For a general non-trivial character χ, one can prove that $L_\chi(s)$ has an analytic continuation to the whole complex plane, and it satisfies a certain functional equation. A similar statement is true for $L_{\chi_0}(s)$. The only difference in this case is that $L_{\chi_0}(s)$ has a simple pole at point $s = 1$.

1.2 Modular Forms

In this section we recall those basic definitions and fundamental results about modular forms and related topics that we will use in this thesis. As we do not give any proofs here, this section will be very concise. The interested reader can consult [11] or [21] for details.

1.2.1 Basic Definitions

Let $\mathbb{H}$ denote the upper half-plane

$$\mathbb{H} = \{z = x + iy : \quad x \in \mathbb{R},\ y > 0\}.$$

Let $GL_2^+(\mathbb{R})$ be the multiplicative group of 2×2 matrices with real entries and positive determinant. Then $GL_2^+(\mathbb{R})$ acts on $\mathbb{H}$ as a group of analytic functions

$$\gamma \ : \ z \mapsto \frac{az+b}{cz+d},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}).$$

Let $\mathbb{H}^*$ denote the union of $\mathbb{H}$ and the rational numbers $\mathbb{Q}$ together with a symbol ∞ (or $i\infty$). The rational numbers together with ∞ are called *cusps*.

Let f be an analytic function on $\mathbb{H}$ and k a positive integer. For

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$$

define the *stroke operator* "$|_k$" as

$$(f|_k\gamma)(z) = (det\gamma)^{\frac{k}{2}}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Sometimes, we simply write $f|\gamma$ for $f|_k\gamma$. Note that $(f|\gamma)|\sigma = f|\gamma\sigma$.

Let $\Gamma = SL_2(\mathbb{Z})$ be the multiplicative group of 2×2 matrices with integer entries and determinant 1 and let Γ' be a subgroup of finite index of it. Suppose f is an analytic function on $\mathbb{H}$ such that $f|\gamma = f$ for all $\gamma \in \Gamma'$. Since Γ' has finite index,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^M = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \in \Gamma'$$

for some positive integer M. Hence $f(z+M) = f(z)$ for all $z \in \mathbb{H}$. So f can be expressed as a function of $q_M = e^{\frac{2\pi i z}{M}}$, which we will denote by $\tilde{f}$. More precisely, there is a function $\tilde{f}$ such that

$$f(z) = \tilde{f}(q_M).$$

The function $\tilde{f}$ is analytic in the punctured disc $0 < |q_M| < 1$. If $\tilde{f}$ extends to a meromorphic (resp. an analytic) function at the origin, we say, by abuse of language, that f is *meromorphic* (resp. *analytic) at infinity.* This means that $\tilde{f}$ has a Laurent expansion in the punctured unit disc. Therefore, f has a *Fourier expansion at infinity* in the form of

$$f(z) = \tilde{f}(q_M) = \sum_{n=-\infty}^{\infty} \hat{a}_f(n) q_M^n, \qquad q_M = e^{\frac{2\pi i z}{M}}$$

where $\hat{a}_f(n) = 0$ for all $n \le n_0$ ($n_0 \in \mathbb{Z}$) if f is meromorphic at infinity; and $\hat{a}_f(n) = 0$ for all $n < 0$ if f is analytic at infinity. We say that f *vanishes at infinity* if $\hat{a}_f(n) = 0$ for all $n \le 0$.

Let $\sigma \in \Gamma$. Then $\sigma^{-1}\Gamma'\sigma$ also has finite index in Γ and $(f|\sigma)|\gamma = f|\sigma$ for all $\gamma \in \sigma^{-1}\Gamma'\sigma$. So $f|\sigma$ also has a Fourier expansion at infinity. We say that f is *analytic at the cusps* if $f|\sigma$ is analytic at infinity for all $\sigma \in \Gamma$. We say that f *vanishes at the cusps* if $f|\sigma$ vanishes at infinity for all $\sigma \in \Gamma$.

Now for $N \ge 1$, let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z});\;\; c \equiv 0 \pmod{N} \right\}.$$

Note that $\Gamma_0(N)$ is of finite index in Γ.

A *modular form of weight k and level N* is an analytic function f on $\mathbb{H}$ such that
(i) $f|\gamma = f$ for all $\gamma \in \Gamma_0(N)$;
(ii) f is analytic at the cusps.
Such a modular form is called a *cusp form* if it vanishes at the cusps.

The modular forms of weight k and level N form a finite dimensional vector space $M_k(N)$ and this has a subspace $S_k(N)$ consisting of cusp forms. Note that since $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the same as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in $\Gamma_0(N)$, *(i)* shows that $M_k(N) = \{0\}$ if k is odd. So from now on we assume that k is even.

Also, one can define an inner product called *Petersson inner product* on $S_k(N)$ by

$$\langle f, g\rangle = \iint_{D_0(N)} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}$$

where $D_0(N)$ is a closed simply connected region in $\mathbb{H}$ with the following two properties:

(i) For any $z \in \mathbb{H}$ there is a $\gamma \in \Gamma_0(N)$ and a $z_1 \in D_0(N)$ such that $z = \gamma(z_1)$;

(ii) If $z_1 = \gamma(z_2)$ where $z_1,\ z_2 \in D_0(N)$ and $\gamma \in \Gamma_0(N)$, then z_1 and z_2 are on the boundary of $D_0(N)$. $D_0(N)$ is called a *fundamental domain* for $\Gamma_0(N)$.

1.2.2 L-Function of a Cusp Form

Let $f \in S_k(N)$. Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(N)$, the Fourier expansion of f at infinity is in the form of

$$f(z) = \sum_{n=1}^{\infty} \hat{a}_f(n)e(nz), \quad e(z) = e^{2\pi i z}.$$

Attached to f, we define the *L-function associated to f* by the Dirichlet series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}$$

where $a_f(n) = \dfrac{\hat{a}_f(n)}{n^{\frac{k-1}{2}}}$ for $n = 1, 2, 3, \cdots$.

It can be shown that $L_f(s)$ represents an analytic function for $Re(s) > 1$. This is a consequence of the fact that for any $\epsilon > 0$, $a_f(n) = O(n^\epsilon)$[2] (see Theorem 1.4).

Let

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

This is not an element of Γ unless $N = 1$. However,

$$W_N \Gamma_0(N) W_N^{-1} = \Gamma_0(N).$$

Moreover, $f|W_N^2 = f$. W_N is called the *Atkin-Lehner involution.* Note that since $f \mapsto f|W_N$ defines a self-inverse linear operator on $S_k(N)$, it decomposes the space of

[2] This means that there exists a constant C such that $|a_f(n)| \le Cn^\epsilon$.

cusp forms $S_k(N)$ to two complementary subspaces corresponding to the eigenvalues ± 1. Set

$$S_k^+(N) = \left\{ f \in S_k(N); \quad f|W_N = (-1)^{\frac{k}{2}} f \right\},$$

$$S_k^-(N) = \left\{ f \in S_k(N); \quad f|W_N = (-1)^{\frac{k}{2}+1} f \right\},$$

and notice that $S_k(N) = S_k^+(N) \oplus S_k^-(N)$. The following Theorem of Hecke guarantees the analytic continuation of $L_f(s)$ for $f \in S_k^{\pm}(N)$.

Theorem 1.1 (Hecke) *Let $f \in S_k^{\pm}(N)$. Then $L_f(s)$ extends to an entire function and $\Lambda_f(s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) L_f(s)$ satisfies the following functional equation*

$$\Lambda_f(s) = \pm \Lambda_f(1-s).$$

The *root number* of $L_f(s)$ is the sign appearing in the functional equation of $L_f(s)$.

Corollary 1.2 *Let $f \in S_k(N)$. Then $L_f(s)$ extends to an entire function.*

Note Our definition of $S_k^+(N)$ and $S_k^-(N)$ is slightly different from the conventional ones that denote them as subspaces corresponding to the eigenvalues $+1$ and -1 for operator W_N, so for $\frac{k}{2}$ odd, our $S_k^{\pm}(N)$ is the conventional $S_k^{\mp}(N)$. In our notation $S_k^{\pm}(N)$ is the set of cusp forms whose L-functions have root number ± 1, respectively.

Let χ be a Dirichlet character (mod q). The *twisted L-function* associated to $f \in S_k(N)$ and χ is defined by the absolutely convergent series

$$L_{f,\chi}(s) = \sum_{n=1}^{\infty} \frac{a_f(n)\chi(n)}{n^s}$$

for $Re(s) > 1$. One can show that if χ is *primitive*[3], then

$$f_\chi(z) = \sum_{n=1}^{\infty} a_f(n)\chi(n) e^{2\pi i n z}$$

is a cusp form of weight k and level $q^2 N$. So in this case, the twisted L-function is the same as the L-function associated to the cusp form f_χ, and hence it has an analytic continuation to the whole complex plane.

[3] This means that the period of χ is exactly q.

1.2.3 Hecke Operators

Let $f \in M_k(N)$. Let p and q be primes such that $p \nmid N$ and $q \mid N$.[4] The *Hecke operators* T_p and U_q are defined by

$$f \mid T_p = p^{\frac{k}{2}-1}\left[f \mid \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{e=0}^{p-1} f \mid \begin{pmatrix} 1 & e \\ 0 & p \end{pmatrix} \right],$$

$$f \mid U_q = q^{\frac{k}{2}-1}\left[\sum_{e=0}^{q-1} f \mid \begin{pmatrix} 1 & e \\ 0 & q \end{pmatrix} \right].$$

We can show that $f \mid T_p$, $f \mid U_q$ are also modular forms of weight k and level N, and furthermore they are cusp forms if f is a cusp form.

Let $f \in S_k(N)$. We will say that f is an *eigenform* if f is an eigenvector for all the Hecke operators $\{T_p\ (p \nmid N),\ U_q\ (q \mid N)\}$. The following theorem gives the main property of eigenforms.

Theorem 1.3 (Hecke) *The following conditions are equivalent.*

(i) f is an eigenform and $a_f(1) = 1$.

(ii) Coefficients $a_f(n)$ satisfy the following three properties:

(a) They are multiplicative, i.e., if g.c.d.$(m, n) = 1$, *then* $a_f(mn) = a_f(m)a_f(n)$;

(b) For $q \mid N$, $a_f(q^l) = a_f(q)^l$;

(c) For $p \nmid N$, $a_f(p^l) = a_f(p)a_f(p^{l-1}) - a_f(p^{l-2})$.

(iii) $L_f(s)$ has a product of the form

$$L_f(s) = \prod_{q \mid N} \left(1 - a_f(q)q^{-s}\right)^{-1} \prod_{p \nmid N} \left(1 - a_f(p)p^{-s} + p^{-2s}\right)^{-1},$$

which converges absolutely for Re$(s) > 1$.

We call the product given in part *(iii)* of the above theorem an *Euler product.* Also any f satisfying the above equivalent conditions is called a *normalized* eigenform. It can be proved that if f is an eigenform, then $a_f(1) \neq 0$. So we can always assume that an eigenform f is normalized.

[4]Here $a \mid b$ means that a is a divisor of b and $a \nmid b$ means that a is not a divisor of b.

The coefficients of modular forms satisfy some important inequalities. The following statement, known as the *Ramanujan-Petersson Conjecture*, gives the best possible bounds for the coefficients of cusp forms.

Theorem 1.4 (Deligne) *(i) If f is a normalized eigenform, then*

$$|a_f(n)| \le \boldsymbol{d}(n)$$

where $\boldsymbol{d}(n)$ is the number of the divisors of n.

(ii) If f is a cusp form, then for any $\epsilon > 0$,

$$a_f(n) \ll n^{\epsilon}.^{5}$$

Now suppose f is an eigenform. From the above inequality it follows that if $p \nmid N$, then $a_f(p)$ can be written in the form of

$$a_f(p) = \epsilon_p + \bar{\epsilon}_p$$

where $\epsilon_p \in \mathbb{C}$ and $|\epsilon_p| = 1$. In fact, ϵ_p and $\bar{\epsilon}_p$ are the roots of the quadratic equation $1 - a_f(p)x + x^2 = 0$.

Corollary 1.5 *If f is a normalized eigenform, then its L-function has the following Euler product, valid for $Re(s) > 1$,*

$$L_f(s) = \prod_{p|N}(1 - a_f(p)p^{-s})^{-1} \prod_{p\nmid N}(1 - \epsilon_p p^{-s})^{-1}(1 - \bar{\epsilon}_p p^{-s})^{-1}.$$

Inspired by the above theorems we may think of finding a basis for $S_k(N)$ consisting of eigenforms for all the operators $\{T_p \ (p \nmid N),\ U_q \ (q \mid N),\ W_N\}$. We can show that there exists a basis for $S_k(N)$ consisting of eigenforms for all the operators $\{T_p \ (p \nmid N)\}$ and the operator W_N (see [5], Lemma 27). The existence of such a basis is the consequence of the fact that $\{T_p \ (p \nmid N),\ W_N\}$ form a commuting family of Hermitian linear operators (with respect to the Petersson inner product) and therefore from a theorem of linear algebra (see [8], p. 207, Theorem 8) the space of cusp forms is diagonalizable under these operators. Unfortunately the operators $\{U_q \ (q \mid N)\}$ are not

[5] $a_f(n) \ll n^{\epsilon}$ means that there exists a constant $C > 0$ such that $|a_f(n)| \le Cn^{\epsilon}$.

Hermitian for $S_k(N)$ and we can not diagonalize $S_k(N)$ with respect to the operators $\{T_p \ (p \nmid N),\ U_q \ (q \mid N),\ W_N\}$. However, we may find such a basis for a certain subspace of $S_k(N)$.

It can be proved that the Fourier coefficient $a_f(n)$ of a normalized eigenform f is real. This is a consequence of the fact that the operators $\{T_p \ (p \nmid N)\}$ are Hermitian, and the fact that the coefficients $a_f(q)$ $(q \mid N)$ are real (see [5], p. 147, Theorem 3).

1.2.4 Oldforms and Newforms

In [5] Atkin and Lehner constructed a subspace of $S_k(N)$ that is diagonalizable under the operators $\{T_p \ (p \nmid N),\ U_q \ (q \mid N),\ W_N\}$. More precisely, they showed that there exists a subspace of $S_k(N)$ whose eigenspaces with respect to the Hecke operators $\{T_p \ (p \nmid N)\}$ are one dimensional. We call such a property, for a subspace of $S_k(N)$, "*multiplicity one*". Now since the operators $\{U_q \ (q \mid N),\ W_N\}$ commute with the operators $\{T_p \ (p \nmid N)\}$, an eigenform for the operators $\{T_p \ (p \nmid N)\}$ is an eigenform for the operators $\{U_q \ (q \mid N),\ W_N\}$ too.

Let $N' \mid N$ $(N' \neq N)$ and suppose that the $\{g_i\}$ is a basis consisting of eigenforms for the operators $\{T_p \ (p \nmid N')\}$. It can be proved that if d is any divisor of $\frac{N}{N'}$ then $g_i(dz) \in S_k(N)$. Set

$$S_k^{\text{old}}(N) = \text{span}\left\{ g_i(dz) : \text{for any} \quad N' \mid N \ (N' \neq N), \quad d \mid \frac{N}{N'} \right\}.$$

We call $S_k^{\text{old}}(N)$ the space of *oldforms*. Its orthogonal complement under the Petersson inner product is denoted by $S_k^{\text{new}}(N)$ and the eigenforms in this space are called *newforms*. So we have

$$S_k(N) = S_k^{\text{old}}(N) \oplus S_k^{\text{new}}(N).$$

Since the space of newforms has multiplicity one, the set of normalized newforms of weight k and level N is uniquely determined. We denote it by $\mathcal{F}_N$. From the above discussion it is clear that if $f \in \mathcal{F}_N$, $L_f(s)$ satisfies a functional equation and has an Euler product on the half-plane $Re(s) > 1$.

1.3 This Thesis

The second chapter of this thesis gives a detailed exposition of Rankin's classical work on the convolution of two modular L-functions. For the modular L-functions $L_f(s) = \sum_{n=1}^{\infty} a_f(n)n^{-s}$ and $L_g(s) = \sum_{n=1}^{\infty} a_g(n)n^{-s}$, let

$$L(f \times g, s) = \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s}.$$

This is called the *Rankin-Selberg convolution* of $L_f(s)$ and $L_g(s)$. In [19] Rankin established the analytic continuation and the functional equation of $L(f \times g, s)$. A detailed proof of Rankin's Theorem is given in Chapter 2 (see Theorem 2.12).

Rankin's Theorem has numerous number theoretic applications. In [18], Rankin used this theorem to prove the non-vanishing of the modular L-function associated to the discriminant function

$$\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24}$$

on the line $Re(s) = 1$. In fact, Rankin's argument establishes the non-vanishing of L-functions associated to eigenforms for the points on the line $Re(s) = 1$, except the point $s = 1$. In [17], Ogg proved that the same result is true for $s = 1$. Moreover, he showed that if $\langle f, g \rangle = 0$, then $L(f \times g, 1) \neq 0$.

In Chapter 3, inspired by Ogg's theorem on non-vanishing of Rankin-Selberg convolutions at the point $s = 1$, we prove two general non-vanishing theorems for the convolution of two general Dirichlet series. One of the main themes of Chapter 3 is that the existence of analytic continuation for certain Dirichlet series is closely related to the problem of non-vanishing of L-functions on the line $Re(s) = 1$. To describe our results, we need the following four definitions.

(i) Let $\bar{\mathcal{S}}$ denote the class of Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_F(n)n^{-s}$ with the following properties:

(a) For $Re(s) > 1$, $F(s)$ has the Euler product

$$F(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}\right);$$

(b) For any $\epsilon > 0$, $a_F(n) = O(n^\epsilon)$;

(c) $F(s)$ has an analytic continuation to the line $Re(s) = 1$, except for a possible pole at $s = 1$.

(ii) For $F \in \bar{\mathcal{S}}$, we define

$$\bar{F}(s) = \overline{F(\bar{s})} = \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^s}.$$

(iii) For $F,\ G \in \bar{\mathcal{S}}$, the *Euler product convolution* of F and G is defined as

$$(F \otimes G)(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{k b_F(p^k)\overline{b_G(p^k)}}{p^{ks}}\right).$$

(iv) For $F \in \bar{\mathcal{S}}$ and $\sigma_0 \leq 1$, we say F is $\otimes$-*simple* in $Re(s) > \sigma_0$ (resp. $Re(s) \geq \sigma_0$), if $F \otimes F$ has an analytic continuation to $Re(s) > \sigma_0$ (resp. $Re(s) \geq \sigma_0$), except for a possible simple pole at $s = 1$.

Section 3.3 deals with the non-vanishing of the Euler product convolution $F \otimes G$ on the line $Re(s) = 1$ $(s \neq 1)$. We prove the following.

Theorem 3.10 *Let $F,\ G \in \bar{\mathcal{S}}$ be $\otimes$-simple in $Re(s) \geq 1$ and let $t \neq 0$. Then*

(i) $(F \otimes F)(1 + it) \neq 0$.

(ii) If $F = \bar{F}$, $G = \bar{G}$, and if $F \otimes G$ has an analytic continuation to the line $Re(s) = 1$, then $(F \otimes G)(1 + it) \neq 0$.

In Section 3.4 we derive a similar non-vanishing result for $s = 1$. We prove the following.

Theorem 3.15 *Let $\sigma_0 < 1$, and assume the following:*

(i) $F \otimes F$ and $G \otimes G$ can be extended analytically to the half-plane $Re(s) > \sigma_0$, except for a (possible) simple pole at $s = 1$;

(ii) $F \otimes G$ has an analytic continuation to the half-plane $Re(s) > \sigma_0$;

(iii) At least one of $F \otimes F$, $G \otimes G$, or $F \otimes G$ has zeros in the half-plane $Re(s) > \sigma_0$. Then $(F \otimes G)(1) \neq 0$.

We will see that the non-vanishing of various L-functions of number theory will be simple consequences of these theorems. In particular, these results establish the non-vanishing of $L_f(s)$, $L_{f,\chi}(s)$ and $L(f \times g, s)$ on the line $Re(s) = 1$. We also observe that, for a non-real character χ, the non-vanishing of $L_\chi(s)$ on the line $Re(s) = 1$ $(s \neq 1)$ does not follow from Theorem 3.10. In order to deal with this problem, in Section 3.5, inspired by a theorem of Ingham [9], we prove a general non-vanishing theorem for the Dirichlet series with completely multiplicative coefficients. More precisely, we prove the following.

Theorem 3.22 *Let F, $G \in \bar{\mathcal{S}}$ be two Dirichlet series with completely multiplicative coefficients. Also assume the following:*

(i) F and G are $\otimes$-simple in $Re(s) \geq \frac{1}{2}$;

(ii) $F \otimes G$ has an analytic continuation to $Re(s) \geq \frac{1}{2}$;

(iii) $(F \otimes G) \otimes (F \otimes G)$ is analytic for $Re(s) > 1$ and has a pole at $s = 1$.

Then, $(F \otimes G)(1 + it) \neq 0$ for all t.

Note that this theorem will imply the non-vanishing of $L_\chi(s)$ on the line $Re(s) = 1$ $(s \neq 1)$.

Chapter 4 is related to the symmetric square L-function associated to a newform f of level N defined by

$$L(\mathrm{sym}^2 f, s) = \frac{\zeta_N(2s)}{\zeta_N(s)} L(f \times f, s).$$

The non-vanishing of $L(\mathrm{sym}^2 f, s)$ inside the *critical strip* (i.e., the strip $0 \leq Re(s) \leq 1$) is the main focus of this chapter. More specifically, we are interested in the following problem.

Problem *Let s_0 be a point inside the critical strip and $\mathcal{F}_N$ be the set of normalized newforms of weight k and level N. Then, what can we say about the*

$$\#\left\{f \in \mathcal{F}_N : \ L(\mathrm{sym}^2 f, s_0) \neq 0\right\}$$

for large N?

This is a challenging problem. The *Generalized Riemann Hypothesis* predicts that for all $f \in \mathcal{F}_N$ and $\frac{1}{2} < Re(s_0) < 1$, $L(\mathrm{sym}^2 f, s_0) \neq 0$. However, we are very far from a proof of this conjecture. Probably the most interesting known result related to the above problem is a result of Kohnen and Sengupta [12]. They prove that for any given s_0 inside the critical strip, we can find an integer k_0 such that, for all $k > k_0$, there exists a newform f for which $L(\mathrm{sym}^2 f, s_0) \neq 0$. This is a nice result, however, it does not directly address the above problem, since in our case the weight is fixed while Kohnen and Sengupta vary the weights.

In the final chapter of this thesis, we prove a partial result related to the above mentioned problem. More precisely, for a fixed point s_0 inside the strip $1-\frac{1}{46} < Re(s) < 1$, we find a lower bound in terms of prime N for the number of weight 2 newforms f for which $L(\mathrm{sym}^2 f, s_0) \neq 0$. The main step in the proof of such a result is establishing an upper bound for the mean values of the symmetric square L-functions in the critical strip.

¿From now on let $\mathcal{F}_N$ be the set of newforms of weight 2 and prime level N. In Section 4.4, we derive an upper bound for the following mean square of symmetric square L-functions

$$\sum_{f\in\mathcal{F}_N} |L(\mathrm{sym}^2 f, s_0)|^2.$$

In [10], Iwaniec and Michel proved such an upper bound in the case of $Re(s_0) = \frac{1}{2}$. We closely follow their approach, and show that a similar result is true for a point inside the critical strip. We prove the following.

Theorem 4.1 *Let s_0 be a point in the strip $\frac{3}{4} \leq \sigma_0 = Re(s_0) \leq 1$. Then,*

$$\sum_{f\in\mathcal{F}_N} |L(\mathrm{sym}^2 f, s_0)|^2 \ll |s_0|^{9+\frac{6}{\epsilon}} N^{1+\epsilon}$$

for any $\epsilon > 0$. The implied constant depends only on ϵ.

In Section 4.5, we combine this theorem with a known result about the values of the symmetric square L-functions on average to find a lower bound in terms of N for the number of newforms f for which $L(\mathrm{sym}^2 f, s_0)$. We have the following.

Theorem 4.10 *Let N be a prime number and let $s_0 = \sigma_0 + it_0$ with $1-\frac{1}{46} < \sigma_0 < 1$. Then for any $\epsilon > 0$, there are positive constants $C_{s_0,\epsilon}$ and $C'_{s_0,\epsilon}$ (depending only on s_0 and ϵ), such that for any prime $N > C'_{s_0,\epsilon}$, there exist at least $C_{s_0,\epsilon}N^{1-\epsilon}$ newforms f of weight 2 and level N for which $L(\mathrm{sym}^2 f, s_0) \neq 0$.*

Finally, the following will be a simple corollary of our result.

Corollary 4.11 *For any $s_0 = \sigma_0 + it_0$ with $1-\frac{1}{46} < \sigma_0 < 1$, there are infinitely many symmetric square L-functions associated to newforms f such that $L(\mathrm{sym}^2 f, s_0) \neq 0$.*

Chapter 2

Rankin-Selberg Convolution

In 1939 Rankin published two papers about modular forms and the behavior of their coefficients ([18], [19]). These papers have played a very influential role in the history of modular forms. In his second paper, he introduced the notion of the convolution of two L-functions associated to modular forms, known as Rankin-Selberg convolution. The main goal of this chapter is to give an exposition of Rankin's work. Section 2.2 is devoted to this. In order to study the Rankin-Selberg convolution, we need some basic properties of the Epstein zeta-function. We will discuss these properties in the first section. Also in the last section we will introduce the symmetric square L-function associated to an eigenform, as we need it in the final chapter of this thesis.

2.1 Epstein Zeta-Function

In this section we introduce the Epstein zeta-function and will establish its basic properties. We will use these results in the next section. Our account will be brief.

Definition 2.1 *For any $z = x + iy \in \mathbb{H}$ and for $s = \sigma + it \in \mathbb{C}$, we define the Epstein zeta-function by*

$$E(z, s) = {\sum_{m,n}}' \frac{1}{|mz + n|^{2s}}$$

where the dash means that m and n run through all integer pairs except (0, 0).

It can be proved that for any $z \in \mathbb{H}$, the above double series is absolutely and uniformly convergent in the half-plane $Re(s) > 1$, and therefore $E(z, s)$ is an analytic function of s on this half-plane (see [4], p. 7).

Our goal here is to prove that the Epstein zeta-function has an analytic continuation and it satisfies a functional equation. Both of these statements are consequences of the transformation property of the following theta-function.

Definition 2.2 *For $\omega > 0$ and $z = x + iy \in \mathbb{H}$, the theta-function $\Theta(\omega)$ is defined by the following infinite sum*

$$\Theta(\omega) = \Theta(z, \omega) = \sum_{m,n}{}' \exp\left\{-\frac{\pi\omega}{y}|mz+n|^2\right\}.$$

Here dash has the same meaning as in the definition of $E(z, s)$.

The first target here is to establish the transformation property of $\Theta(\omega)$. To do this, first we recall some facts about the Fourier transform. For simplicity, we set $e(z) = e^{2\pi i z}$.

Definition 2.3 *Let $f : \mathbb{R}^n \to \mathbb{C}$ be bounded, smooth (i.e., all partial derivatives exist and are continuous), and rapidly decreasing (i.e., for any N, $|\mathbf{x}|^N f(\mathbf{x})$ tends to zero when $|\mathbf{x}|$ goes to infinity). The Fourier transform of f is defined by*

$$\hat{f}(\mathbf{y}) = \int_{\mathbb{R}^n} e(-\mathbf{x}^t\mathbf{y}) f(\mathbf{x}) d\mathbf{x}.$$

Here, $\mathbf{x} = (x_1, ..., x_n)^t$, $\mathbf{y} = (y_1, ..., y_n)^t$, $\mathbf{x}^t\mathbf{y} = \sum_{j=1}^{n} x_j y_j$, $|\mathbf{x}| = (\mathbf{x}^t\mathbf{x})^{\frac{1}{2}}$, $d\mathbf{x} = \prod_{j=1}^{n} dx_j$ *and "t" stands for transposition.*

It can be proved that for $f(\mathbf{x}) = e^{-\pi\mathbf{x}^t\mathbf{x}}$ we have $\hat{f} = f$ (see [11], p. 83). Recall that throughout this chapter ω is a positive real number.

Lemma 2.4 *Let A be a real symmetric matrix of size n with positive eigenvalues, and let*

$$g(\mathbf{x}) = e\left(\frac{i}{2}\omega\mathbf{x}^t A\mathbf{x}\right) = e^{-\pi\omega\mathbf{x}^t A\mathbf{x}}.$$

Then we have

$$\hat{g}(\mathbf{y}) = |A|^{-\frac{1}{2}} \left(\frac{1}{\omega}\right)^{\frac{n}{2}} e\left(\frac{i}{2\omega}\mathbf{y}^t A^{-1}\mathbf{y}\right) = |A|^{-\frac{1}{2}} \left(\frac{1}{\omega}\right)^{\frac{n}{2}} e^{-\frac{\pi}{\omega}\mathbf{y}^t A^{-1}\mathbf{y}}.$$

Here, $|A|$ is the determinant of A.

Proof By the principal axis theorem (see [8], p. 323, Theorem 4), there exists an orthogonal matrix U such that

$$A = U^t D U$$

where $D = \text{diag}[\lambda_1, ..., \lambda_n]$ is a diagonal matrix and λ_i's are the eigenvalues of A. Let

$$B = \text{diag}[\sqrt{\lambda_1}, ..., \sqrt{\lambda_n}]U = (b_{ij})_{n\times n}.$$

B is invertible and $A = B^t B$. Consider the change of variable $\mathbf{u} = \omega^{\frac{1}{2}} B\mathbf{x}$, and let $\mathbf{v} = \omega^{-\frac{1}{2}}(B^t)^{-1}\mathbf{y}$. We have the following

$$\mathbf{u}^t\mathbf{u} = \omega\mathbf{x}^t A\mathbf{x}, \quad \mathbf{v}^t\mathbf{v} = \frac{1}{\omega}\mathbf{y}^t A^{-1}\mathbf{y}, \quad \mathbf{x}^t\mathbf{y} = \mathbf{u}^t\mathbf{v}.$$

Also for the Jacobian matrix $\mathcal{J}$ we have

$$\mathcal{J} = \left(\frac{\partial u_i}{\partial x_j}\right)_{n\times n} = \left(\omega^{\frac{1}{2}} b_{ij}\right)_{n\times n} = \omega^{\frac{1}{2}} B,$$

and therefore

$$d\mathbf{u} = |\mathcal{J}| d\mathbf{x} = \omega^{\frac{n}{2}} |B| d\mathbf{x} = \omega^{\frac{n}{2}} |A|^{\frac{1}{2}} d\mathbf{x}.$$

Applying this change of variable in the Fourier transform of g yields

$$\begin{aligned}
\hat{g}(\mathbf{y}) &= \int_{\mathbb{R}^n} e(-\mathbf{x}^t\mathbf{y}) e^{-\pi\omega\mathbf{x}^t A\mathbf{x}} d\mathbf{x} \\
&= |A|^{-\frac{1}{2}} \left(\frac{1}{\omega}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e(-\mathbf{u}^t\mathbf{v}) e^{-\pi\mathbf{u}^t\mathbf{u}} d\mathbf{u} \\
&= |A|^{-\frac{1}{2}} \left(\frac{1}{\omega}\right)^{\frac{n}{2}} \hat{f}(\mathbf{v}) \\
&= |A|^{-\frac{1}{2}} \left(\frac{1}{\omega}\right)^{\frac{n}{2}} e^{-\pi\mathbf{v}^t\mathbf{v}} \\
&= |A|^{-\frac{1}{2}} \left(\frac{1}{\omega}\right)^{\frac{n}{2}} e^{-\frac{\pi}{\omega}\mathbf{y}^t A^{-1}\mathbf{y}}.
\end{aligned}$$

The proof is complete. □

Proposition 2.5 *The theta-function $\Theta(\omega)$ satisfies the following transformation property*

$$1+\Theta(\omega)=\frac{1}{\omega}\left(1+\Theta\left(\frac{1}{\omega}\right)\right).$$

Proof In Lemma 2.4 put

$$A=\begin{pmatrix} \frac{|z|^2}{y} & \frac{x}{y} \\ \frac{x}{y} & \frac{1}{y} \end{pmatrix}$$

where $z=x+iy\in\mathbb{H}$. A has positive eigenvalues and we have

$$A^{-1}=\begin{pmatrix} \frac{1}{y} & -\frac{x}{y} \\ -\frac{x}{y} & \frac{|z|^2}{y} \end{pmatrix},\quad n=2,\quad |A|=1,$$

and so

$$g(\mathbf{x})=e^{-\pi\omega\mathbf{x}^tA\mathbf{x}}=e^{-\frac{\pi\omega}{y}|x_1z+x_2|^2},$$

$$\hat{g}(\mathbf{y})=\frac{1}{\omega}e^{-\frac{\pi}{\omega}\mathbf{y}^tA^{-1}\mathbf{y}}=\frac{1}{\omega}e^{-\frac{\pi}{y\omega}|y_1-y_2z|^2}.$$

By applying the *Poisson summation formula*, i.e.,

$$\sum_{\mathbf{m}\in\mathbb{Z}^2}g(\mathbf{m})=\sum_{\mathbf{m}\in\mathbb{Z}^2}\hat{g}(\mathbf{m}),$$

(see [11], p. 83), we have

$$\sum_{m,n}e^{-\frac{\pi\omega}{y}|mz+n|^2}=\frac{1}{\omega}\sum_{m,n}e^{-\frac{\pi}{y\omega}|m-nz|^2}$$

or

$$1+\Theta(\omega)=\frac{1}{\omega}\left(1+\Theta\left(\frac{1}{\omega}\right)\right).$$

The proof is complete. □

In the sequel, we also need to know how $\Theta(\omega)$ behaves at infinity. It can be proved that $\Theta(\omega)$ has exponential decay. More precisely, by approximation methods, one can show that for $-1\le Re(z)\le 1$,

$$\Theta(\omega)\ll\left(1+\omega^{-1}+y^{\frac{1}{2}}\omega^{-\frac{1}{2}}+y^{-\frac{1}{2}}\omega^{-\frac{1}{2}}\right)\left(e^{-\frac{\pi y\omega}{2}}+e^{-\frac{\pi\omega}{2y}}\right). \tag{2.1}$$

(See [19], p. 359, Lemma 3).

Now we are ready to prove the main result of this section.

Proposition 2.6 *(i) The Epstein zeta-function can be analytically continued to the whole complex plane, except for a simple pole at $s = 1$ with residue $\dfrac{\pi}{y}$.*

(ii) Put

$$\xi(z,s) = \left(\frac{\pi}{y}\right)^{-s} \Gamma(s)E(z,s).$$

We have the following integral representation for $\xi(z,s)$

$$\xi(z,s) = \int_1^{\infty} \Theta(\omega)(\omega^{s-1} + \omega^{-s})d\omega + \frac{1}{s(s-1)}$$

and so, $\xi(z,s)$ is analytic everywhere, except for simple poles at $s = 0, 1$ with residue 1.

(iii) $\xi(z,s)$ is unchanged under the replacing of s by $1-s$. This means that

$$\xi(z,s) = \xi(z,1-s).$$

In other words, the Epstein zeta-function satisfies the following functional equation

$$\left(\frac{\pi}{y}\right)^{-s} \Gamma(s)E(z,s) = \left(\frac{\pi}{y}\right)^{s-1} \Gamma(1-s)E(z,1-s).$$

Proof For $Re(s) > 0$, we have

$$\Gamma(s) = \int_0^{\infty} e^{-u}u^{s-1}du.$$

We apply the change of variable $u \mapsto \frac{\pi}{y}|mz+n|^2\omega$, to get

$$\Gamma(s) = \left(\frac{\pi}{y}\right)^{s} |mz+n|^{2s} \int_0^{\infty} e^{-\frac{\pi\omega}{y}|mz+n|^2}\omega^{s-1}d\omega,$$

or

$$\left(\frac{\pi}{y}\right)^{-s} |mz+n|^{-2s}\Gamma(s) = \int_0^{\infty} e^{-\frac{\pi\omega}{y}|mz+n|^2}\omega^{s-1}d\omega.$$

This implies

$$\begin{aligned}
\xi(z,s) &= \left(\frac{\pi}{y}\right)^{-s}\Gamma(s)E(z,s)\\
&= \left(\frac{\pi}{y}\right)^{-s}\Gamma(s)\sum_{m,n}{}'\frac{1}{|mz+n|^{2s}}\\
&= \sum_{m,n}{}'\left(\frac{\pi}{y}\right)^{-s}\Gamma(s)|mz+n|^{-2s}\\
&= \sum_{m,n}{}'\int_0^\infty \exp\left\{-\frac{\pi\omega}{y}|mz+n|^2\right\}\omega^{s-1}d\omega.
\end{aligned}$$

Now note that the inequality (2.1) allows us to interchange the order of summation and integration. So

$$\begin{aligned}
\xi(z,s) &= \int_0^\infty \sum_{m,n}{}'\exp\left\{-\frac{\pi\omega}{y}|mz+n|^2\right\}\omega^{s-1}d\omega\\
&= \int_0^\infty \Theta(\omega)\omega^{s-1}d\omega\\
&= \int_0^1 \Theta(\omega)\omega^{s-1}d\omega+\int_1^\infty \Theta(\omega)\omega^{s-1}d\omega.
\end{aligned}$$

Changing variable $u \mapsto \frac{1}{\omega}$ and applying the transformation property of Proposition 2.5 yield

$$\begin{aligned}
\xi(z,s) &= \int_1^\infty \Theta(\omega)\omega^{s-1}d\omega+\int_\infty^1 \Theta\left(\frac{1}{u}\right)\left(\frac{1}{u}\right)^{s-1}\left(-\frac{1}{u^2}\right)du\\
&= \int_1^\infty \Theta(\omega)\omega^{s-1}d\omega+\int_1^\infty \left\{\omega\left(1+\Theta(\omega)\right)-1\right\}\left(\frac{1}{\omega}\right)^{s+1}d\omega\\
&= \int_1^\infty \Theta(\omega)\omega^{s-1}d\omega+\int_1^\infty \Theta(\omega)\omega^{-s}d\omega+\int_1^\infty \left(\omega^{-s}-\omega^{-s-1}\right)d\omega\\
&= \int_1^\infty \Theta(\omega)\left(\omega^{s-1}+\omega^{-s}\right)d\omega+\frac{1}{s(s-1)}. \qquad (2.2)
\end{aligned}$$

Note that the inequality (2.1) also shows that

$$\int_1^\infty \left|\Theta(\omega)(\omega^{s-1}+\omega^{-s})\right|d\omega$$

$$\ll \int_1^\infty \left(1+\omega^{-1}+y^{\frac{1}{2}}\omega^{-\frac{1}{2}}+y^{-\frac{1}{2}}\omega^{-\frac{1}{2}}\right)\left(e^{-\frac{\pi y\omega}{2}}+e^{-\frac{\pi\omega}{2y}}\right)\left(\omega^{\sigma-1}+\omega^{-\sigma}\right)d\omega.$$

After expanding the right-hand side, we come to a finite sum of integrals in the form of

$$\int_1^\infty e^{-a\omega}\omega^b d\omega$$

where $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$. Since these integrals are convergent, the first summand on the right-hand side of (2.2) is an entire function of s. This proves *(ii)*.

The identity (2.2) also proves *(iii)*, because the right-hand side of (2.2) is invariant under the replacing of s with $1-s$.

To prove *(i)*, note that by *(ii)* the only possible poles for $E(z,s)$ are $s=0,1$. At $s=0$ since both $\Gamma(s)$ and $\xi(z,s)$ have simple poles with residue 1, $E(z,s)$ is analytic and $E(z,0)=1$. At $s=1$, $\Gamma(s)$ has a value of 1 and $\xi(z,s)$ has a simple pole with residue 1. Therefore $E(z,s)$ has a simple pole with residue $\frac{\pi}{y}$.
This completes the proof. □

2.2 Rankin-Selberg Convolution

Let $z=x+iy$ be a point in the upper half-plane $\mathbb{H}$, and let $s=\sigma+it$ be a point in the complex plane $\mathbb{C}$. Let

$$f(z)=\sum_{n=1}^\infty \hat{a}_f(n)e^{2\pi inz}$$

and

$$g(z)=\sum_{n=1}^\infty \hat{a}_g(n)e^{2\pi inz}$$

be cusp forms of weight k and level N. We set

$$\delta(f,g)=y^{k-2}f(z)\overline{g(z)}.$$

Recall that for $Re(s)>1$, the L-functions attached to f and g are defined by

$$L_f(s)=\sum_{n=1}^\infty \frac{a_f(n)}{n^s}$$

and

$$L_g(s) = \sum_{n=1}^{\infty} \frac{a_g(n)}{n^s}$$

where

$$a_f(n) = \frac{\hat{a}_f(n)}{n^{\frac{k-1}{2}}}, \ a_g(n) = \frac{\hat{a}_g(n)}{n^{\frac{k-1}{2}}}$$

for $n = 1, 2, 3, \cdots$.

Definition 2.7 *The Rankin-Selberg convolution of $L_f(s)$ and $L_g(s)$ is defined by*

$$L(f \times g, s) = \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s}.$$

The modified Rankin-Selberg convolution of $L_f(s)$ and $L_g(s)$ is defined by

$$L(f \otimes g, s) = \zeta_N(2s)L(f \times g, s) = \zeta_N(2s) \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s}$$

where $\zeta_N(s) = \sum_{\substack{n=1 \\ \text{g.c.d.}(n,N)=1}}^{\infty} \frac{1}{n^s} = \prod_{p \nmid N} \left(1 - \frac{1}{p^s}\right)^{-1}$ *is the Riemann zeta-function with the Euler p-factors corresponding to $p \mid N$ removed.*

The main goal of this section is to study the analytic properties of $L(f \times g, s)$. We will see that the analytic continuation and the functional equation of the Epstein zeta-function $E(z, s)$ will result in the analytic continuation and the functional equation for the Rankin-Selberg convolution $L(f \times g, s)$.

In Proposition 2.9 we will relate the Rankin-Selberg convolution $L(f \times g, s)$ to a double integral on a certain region of the upper half-plane. To do this we need the following lemma.

Lemma 2.8 *For any fixed $y > 0$,*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(z)\overline{g(z)}dx = \sum_{n=1}^{\infty} \hat{a}_f(n)\overline{\hat{a}_g(n)}e^{-4\pi ny}.$$

Proof We have

$$\begin{aligned}
\int_{-\frac{1}{2}}^{\frac{1}{2}} f(z)\overline{g(z)}dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} \hat{a}_f(m) e^{2\pi i m(x+iy)} \overline{\sum_{n=1}^{\infty} \hat{a}_g(n) e^{2\pi i n(x+iy)}} \right) dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \hat{a}_f(m)\overline{\hat{a}_g(n)} e^{2\pi i(m-n)x} e^{-2\pi(m+n)y} \right) dx.
\end{aligned}$$

Interchanging the order of summation and integration yields

$$\begin{aligned}
\int_{-\frac{1}{2}}^{\frac{1}{2}} f(z)\overline{g(z)}dx &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\hat{a}_f(m)\overline{\hat{a}_g(n)} e^{-2\pi(m+n)y} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i(m-n)x} dx \right) \\
&= \sum_{n=1}^{\infty} \hat{a}_f(n)\overline{\hat{a}_g(n)} e^{-4\pi n y}.
\end{aligned}$$

The proof is complete. $\square$

Proposition 2.9 *For* $Re(s) > 1$ *we have the following integral representation for the Rankin-Selberg convolution* $L(f \times g, s)$

$$\begin{aligned}
(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f \times g, s) &= \iint_S y^{s+k-2} f(z)\overline{g(z)} dx dy \\
&= \iint_S y^s \delta(f,g) dx dy
\end{aligned}$$

where S *is the strip* $|x| \le \frac{1}{2}$ *and* $y > 0$.

Proof We have

$$\begin{aligned}
(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f \times g, s) &= (4\pi)^{-s-k+1}\Gamma(s+k-1) \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s} \\
&= \sum_{n=1}^{\infty} \left\{ \frac{\hat{a}_f(n)\overline{\hat{a}_g(n)}}{n^{k-1}} \frac{(4\pi)^{-s-k+1}}{n^s} \Gamma(s+k-1) \right\} \\
&= \sum_{n=1}^{\infty} \left\{ \hat{a}_f(n)\overline{\hat{a}_g(n)} (4\pi n)^{-s-k+1} \Gamma(s+k-1) \right\}.
\end{aligned}$$

Note that by the change of variable $t \mapsto 4\pi n y$, $\Gamma(s+k-1)$ can be written as

$$\Gamma(s+k-1) = (4\pi n)^{s+k-1} \int_0^{\infty} e^{-4\pi n y} y^{s+k-2} dy.$$

So

$$\begin{aligned}(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f\times g,s) &= \sum_{n=1}^{\infty}\left\{\hat{a}_f(n)\overline{\hat{a}_g(n)}\int_0^{\infty} e^{-4\pi ny}y^{s+k-2}\right\}dy \\ &= \int_0^{\infty} y^{s+k-2}\left\{\sum_{n=1}^{\infty}\hat{a}_f(n)\overline{\hat{a}_g(n)}e^{-4\pi ny}\right\}dy.\end{aligned}$$

Now by applying Lemma 2.8 we get

$$\begin{aligned}(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f\times g,s) &= \int_0^{\infty} y^{s+k-2}\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}} f(z)\overline{g(z)}dx\right\}dy \\ &= \iint_S y^{s+k-2}f(z)\overline{g(z)}dxdy \\ &= \iint_S y^s\delta(f,g)dxdy.\end{aligned}$$

This completes the proof. □

Our next step is to rewrite the double integral in the statement of the previous proposition as a new integral on a fundamental domain for $\Gamma_0(N)$.

Lemma 2.10 *We have*

$$\iint_S y^s\delta(f,g)dxdy = \iint_{D_0(N)} y^s\delta(f,g)F_N(z,s)dxdy$$

where

$$F_N(z,s) = 1+\sum_{m=1}^{\infty}\sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,mN)=1}}^{\infty}\frac{1}{|mNz+n|^{2s}}$$

and $D_0(N)$ is a fundamental domain for $\Gamma_0(N)$.

Proof Let

$$\Gamma_\infty = \{\gamma\in\Gamma:\ \gamma\infty=\infty\} = \left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}:\ b\in\mathbb{Z}\right\}.$$

Γ_∞ is a subgroup of Γ and it is clear that the strip $S=\{(x,y):\ |x|\le\frac{1}{2},\ y>0\}$ is a fundamental domain for Γ_∞. For any two matrices $\gamma=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ and $\gamma'=\left(\begin{smallmatrix}a' & b'\\ c' & d'\end{smallmatrix}\right)$ in $GL_2(\mathbb{Z})$, the right cosets $\Gamma_\infty\gamma$ and $\Gamma_\infty\gamma'$ are equal if and only if $(c,d) = \pm(c',d')$. So the right cosets of Γ_∞ in $\Gamma_0(N)$ are in one to one correspondence with the pairs (c,d). Therefore we can choose a set of representative $\mathcal{T}$ for the right cosets of Γ_∞ in $\Gamma_0(N)$ as follows:

$$\mathcal{T} = \{(0,1)\} \cup \{(c,d): \ c > 0,\ N|c,\ (c,d) = 1\}.$$

We claim that for any pair (c,d) in $\mathcal{T}$, there is a unique transformation

$$\gamma_{c,d}: z_1 \longrightarrow z = \frac{az_1 + b}{cz_1 + d}$$

that maps $D_0(N)$ into S. This is true for the pair $(0,1)$. For other pairs in $\mathcal{T}$, note that since $\infty \in D_0(N)$,

$$\left|\frac{a}{c}\right| = |\gamma_{c,d}(\infty)| \leq \frac{1}{2}.$$

This shows that $c \geq 2$; and since $ad - bc = 1$, equality holds only if $c = 2$, $a = \pm 1$. We consider two cases.

If $c > 2$, then there is exactly one solution in $a,\ b$ of the equation $ad - bc = 1$ for which $\left|\frac{a}{c}\right| < \frac{1}{2}$. Since $\gamma_{c,d}D_0(N)$ has the unique cusp $\frac{a}{c}$ in S, and this cusp is not on either of the lines $|x| = \frac{1}{2}$, the whole of $\gamma_{c,d}D_0(N)$ lies in S.

If $c = 2$, then $a = \pm 1$. Suppose that, for example, $\gamma_{c,d}$ takes ∞ to the cusp $-\frac{1}{2}$ and takes $D_0(N)$ into S. Then the transformation $\gamma_{c,d}(z_1) + 1$ has the same $c,\ d$ and maps $D_0(N)$ outside S (touching the line $x = \frac{1}{2}$), and therefore corresponds to the other solution. Hence exactly one of the transformations $\gamma_{c,d}(z_1)$ or $\gamma_{c,d}(z_1) + 1$ has the desired property. The claim is proved.

This shows that the strip S can be written as the disjoint union of $\gamma_{c,d}D_0(N)$'s

$$S = \bigcup_{(c,d)\in\mathcal{T}} \gamma_{c,d}D_0(N).$$

Therefore, we have

$$\iint_S y^s \delta(f,g)dxdy = \sum_{(c,d)\in\mathcal{T}} \iint_{\gamma_{c,d}D_0(N)} y^s \delta(f,g)dxdy.$$

Now let $z_1 = x_1 + iy_1$. Changing variable $z_1 \mapsto z = \dfrac{az_1+b}{cz_1+d}$ yields

$$\begin{aligned}
\iint_S y^s \delta(f,g)dxdy &= \sum_{(c,d)\in\mathcal{T}} \iint_{D_0(N)} \left(\frac{y_1}{|cz_1+d|^2}\right)^s \delta(f,g)dx_1dy_1 \\
&= \iint_{D_0(N)} \left(y_1^s \delta(f,g) \sum_{(c,d)\in\mathcal{T}} \frac{1}{|cz_1+d|^{2s}} \right) dx_1dy_1.
\end{aligned}$$

By considering the definition of $\mathcal{T}$ in the last integral, we have

$$\begin{aligned}
\iint_S y^s \delta(f,g)dxdy &= \iint_{D_0(N)} y^s\delta(f,g) \left\{ 1 + \sum_{\substack{c=1 \\ N|c}}^{\infty} \sum_{\substack{d=-\infty \\ \text{g.c.d.}(d,c)=1}}^{\infty} \frac{1}{|cz+d|^{2s}} \right\} dxdy \\
&= \iint_{D_0(N)} y^s\delta(f,g)F_N(z,s)dxdy.
\end{aligned}$$

The proof is complete. □

Now we will show that $F_N(z,s)$ has a representation in terms of the Epstein zeta-function. First we recall the definition of the Möbius function.

The *Möbius function* $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1p_2\cdots p_r,\ p_i \neq p_j \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.11 *We have*

$$2\zeta_N(2s)F_N(z,s) = \sum_{d|N} \frac{\mu(d)}{d^{2s}} E(\frac{N}{d}z, s).$$

Proof The idea is to evaluate the double sum

$$S = \sum_{\substack{m,n \\ \text{g.c.d.}(n,N)=1}}{}' \frac{1}{|mNz+n|^{2s}}$$

in two different ways.

On one hand we have

$$\begin{aligned} S &= 2\sum_{\substack{n=1\\ \text{g.c.d.}(n,N)=1}}^{\infty} \frac{1}{n^{2s}} + \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} \sum_{\substack{n=-\infty\\ \text{g.c.d.}(n,N)=1}}^{\infty} \frac{1}{|mNz+n|^{2s}} \\ &= 2\zeta_N(2s) + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{|mNz+n|^{2s}} \\ &= 2\zeta_N(2s) + 2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \sum_{\substack{n=-\infty\\ \text{g.c.d.}(n,m)=k}}^{\infty} \frac{1}{|mNz+n|^{2s}}. \end{aligned}$$

Note that since g.c.d.$(n,N)=1$, then g.c.d.$(n,m)=$ g.c.d.(n,mN). So

$$\begin{aligned} S &= 2\zeta_N(2s) + 2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \sum_{\substack{n=-\infty\\ \text{g.c.d.}(n,mN)=k}}^{\infty} \frac{1}{|mNz+n|^{2s}} \\ &= 2\zeta_N(2s) + 2\zeta_N(2s)\sum_{m=1}^{\infty} \sum_{\substack{n=-\infty\\ \text{g.c.d.}(n,mN)=1}}^{\infty} \frac{1}{|mNz+n|^{2s}} \\ &= 2\zeta_N(2s)F_N(z,s). \end{aligned}$$

On the other hand by applying the classical identity

$$\sum_{d|n}\mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$$

(see [15], Exercise 1.1.1), we have

$$\begin{aligned} S &= \sum_{m,n}{}' \left\{ \frac{1}{|mNz+n|^{2s}} \sum_{d|\text{g.c.d.}(n,N)} \mu(d) \right\} \\ &= \sum_{d|N} \left\{ \mu(d) \sum_{m,n}{}' \frac{1}{|mNz+n_1 d|^{2s}} \right\} \end{aligned}$$

where $n_1 = \dfrac{n}{d}$. So

$$\begin{aligned} S &= \sum_{d|N} \left\{ \frac{\mu(d)}{d^{2s}} \sum_{m,n_1}{}' \frac{1}{|m\frac{N}{d}z+n_1|^{2s}} \right\} \\ &= \sum_{d|N} \left\{ \frac{\mu(d)}{d^{2s}} E\left(\frac{N}{d}z, s\right) \right\}. \end{aligned}$$

This completes the proof. □

We are ready to prove the main result of this chapter.

Theorem 2.12 (Rankin) *The Rankin-Selberg convolution $L(f \times g, s)$ has the following properties:*

(i) The series

$$L(f \times g, s) = \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s}$$

is absolutely and uniformly convergent for $Re(s) > 1$.

(ii) $L(f \times g, s)$ has a meromorphic continuation to the whole complex plane.

(iii) $L(f \times g, s)$ is analytic at $s = 1$ if $\langle f, g \rangle = 0$. Otherwise, it has a simple pole at point $s = 1$ with the residue

$$\begin{aligned} r &= \frac{12(4\pi)^{k-1}}{N(k-1)!\prod_{p|N}(1-\frac{1}{p})} \iint_{D_0(N)} \delta(f,g)dxdy \\ &= \frac{12(4\pi)^{k-1}}{N(k-1)!\prod_{p|N}(1-\frac{1}{p})} \langle f, g \rangle. \end{aligned}$$

(iv) Let

$$L(f \otimes g, s) = \zeta_N(2s) L(f \times g, s) = \zeta_N(2s) \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s}$$

be the modified Rankin-Selberg convolution and for $Re(s) > 1$, let

$$\begin{aligned} \Phi(s) &= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1) L(f \otimes g, s) \\ &= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)\zeta_N(2s) L(f \times g, s). \end{aligned}$$

Then both $L(f \otimes g, s)$ and $\Phi(s)$ are entire functions if $\langle f, g \rangle = 0$. Otherwise, they are analytic everywhere except that $L(f \otimes g, s)$ has a simple pole at point $s = 1$ and $\Phi(s)$ has simple poles at points $s = 0$ and 1.

(v) If $N = 1$, then the function $\Phi(s)$ is invariant under the replacing of s by $1 - s$, i.e.,

$$\Phi(s) = \Phi(1 - s).$$

Proof *(i)* Suppose that $\sigma = Re(s) \geq 1 + \delta > 1$. By Deligne's bound (see Theorem 1.4), we know that $|a_f(n)|, |a_g(n)| \ll n^{\delta/4}$. So,

$$\begin{aligned}\sum_{n=1}^{\infty} \left| \frac{a_f(n)\overline{a_g(n)}}{n^s} \right| &\ll \sum_{n=1}^{\infty} \frac{n^{\delta/2}}{n^{\sigma}} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta/2}} \\ &< +\infty.\end{aligned}$$

This completes the proof of *(i)*.

(ii) & *(iv)* By Proposition 2.9 and Lemma 2.10, we have

$$\begin{aligned}\Phi(s) &= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)L(f \otimes g, s) \\ &= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\zeta_N(2s)(4\pi)^{s+k-1} \iint_S y^s \delta(f,g) dxdy \\ &= (4\pi)^{k-1} \left(\frac{N}{\pi}\right)^s \Gamma(s)\zeta_N(2s) \iint_{D_0(N)} y^s \delta(f,g) F_N(z,s) dxdy.\end{aligned}$$

Applying Lemma 2.11 in the previous integral yields

$$\begin{aligned}\Phi(s) &= \frac{(4\pi)^{k-1}}{2} \left(\frac{N}{\pi}\right)^s \Gamma(s) \iint_{D_0(N)} y^s \delta(f,g) \sum_{d|N} \left(\frac{\mu(d)}{d^{2s}} E\left(\frac{N}{d}\tau, s\right)\right) dxdy \\ &= \frac{(4\pi)^{k-1}}{2} \iint_{D_0(N)} \delta(f,g) \sum_{d|N} \left(\frac{\mu(d)}{d^s} \left(\frac{Ny}{d\pi}\right)^s \Gamma(s) E\left(\frac{N}{d}\tau, s\right)\right) dxdy.\end{aligned}$$

Finally by (2.2), we obtain

$$\begin{aligned}\Phi(s) &= \frac{(4\pi)^{k-1}}{2} \iint_{D_0(N)} \delta(f,g) \sum_{d|N} \left(\frac{\mu(d)}{d^s} \xi\left(\frac{N}{d}z, s\right)\right) dxdy \\ &= \frac{(4\pi)^{k-1}}{2} \sum_{d|N} \frac{\mu(d)}{d^s} \iint_{D_0(N)} \left(\delta(f,g) \int_1^{\infty} \Theta(\omega)(\omega^{s-1} + \omega^{-s})\right) dxdyd\omega \\ &+ \frac{(4\pi)^{k-1}}{2s(s-1)} \sum_{d|N} \frac{\mu(d)}{d^s} \iint_{D_0(N)} \delta(f,g) dxdy. \qquad (2.3)\end{aligned}$$

Note that by (2.1), the integral in the first summand of the right-hand side of (2.3) is dominated by a finite sum of integrals of the form

$$\iint_{D_0(N)} y^\lambda \delta(f,g) \left(\int_1^\infty e^{-a\omega} \omega^b d\omega \right) dxdy$$

for $\lambda \in \mathbb{R}$. These integrals are all convergent, because f and g vanish at all the cusps of $D_0(N)$. Therefore the first summand in 2.3 is an entire function of s. This proves *(ii)* and *(iv)*.

(iii) If we multiply both sides of (2.3) by $s-1$ and then let $s \to 1^+$, we get

$$\lim_{s\to 1^+} (s-1) \left(\frac{2\pi}{\sqrt{N}} \right)^{-2s} \Gamma(s)\Gamma(s+k-1)\zeta_N(2s)L(f\times g,s)$$
$$= \frac{(4\pi)^{k-1}}{2} \sum_{d|N} \frac{\mu(d)}{d} \iint_{D_0(N)} \delta(f,g)dxdy$$

and therefore

$$\begin{aligned} r &= \operatorname{Res}(L(f\times g,s),1) \\ &= \frac{12(4\pi)^{k-1}}{N(k-1)!\prod_{p|N}(1+\frac{1}{p})} \iint_{D_0(N)} \delta(f,g)dxdy. \end{aligned}$$

This completes the proof of part *(iii)*.

(v) Let $N=1$. We can simplify (2.3) to

$$\begin{aligned} \Phi(s) &= \frac{(4\pi)^{k-1}}{2} \iint_{D_0(1)} \left(\delta(f,g) \int_1^\infty \Theta(\omega)(\omega^{s-1}+\omega^{-s}) \right) dxdyd\omega \\ &+ \frac{(4\pi)^{k-1}}{2s(s-1)} \iint_{D_0(1)} \delta(f,g)dxdy. \end{aligned}$$

At a glance we realize that the right-hand side of this equality is invariant under the replacing of s with $1-s$. Therefore

$$\Phi(s) = \Phi(1-s).$$

In other words, $L(f\times g,s)$ satisfies the following functional equation

$$(2\pi)^{-2s}\Gamma(s)\Gamma(s+k-1)\zeta_N(2s)L(f\times g,s)$$
$$= (2\pi)^{2s-2}\Gamma(1-s)\Gamma(k-s)\zeta_N(2-2s)L(f\times g,1-s).$$

The proof of the theorem is complete. □

In the rest of this section, we will study the Euler product of the Rankin-Selberg convolution of two modular L-functions. Let $f(z) = \sum_{n=1}^{\infty} \hat{a}_f(n) e^{2\pi i n z}$ be a cusp form for $\Gamma_0(N)$, and let $L_f(s) = \sum_{n=1}^{\infty} a_f(n) n^{-s}$ be its associated L-function. From Theorem 1.3, we know that $L_f(s)$ has an Euler product if and only if $f(z)$ be an eigenform. The next proposition will establish the Euler product of the modified Rankin-Selberg convolution of the modular L-functions associated to two eigenforms f and g. To derive the desired Euler product we need the following lemma.

Lemma 2.13 *Let f and g be two normalized eigenforms in $\Gamma_0(N)$, and let*

$$L_f(s) = \prod_{p|N} \left(1 - a_f(p) p^{-s}\right)^{-1} \prod_{p\nmid N} \left(1 - \epsilon_p p^{-s}\right)^{-1} \left(1 - \bar{\epsilon}_p p^{-s}\right)^{-1}$$

and

$$L_g(s) = \prod_{p|N} \left(1 - a_g(p) p^{-s}\right)^{-1} \prod_{p\nmid N} \left(1 - \delta_p p^{-s}\right)^{-1} \left(1 - \bar{\delta}_p p^{-s}\right)^{-1}$$

be their associated L-functions, where $\epsilon_p + \bar{\epsilon}_p = a_f(p)$, $\delta_p + \bar{\delta}_p = a_g(p)$ and $|\epsilon_p| = |\delta_p| = 1$. Then, for $Re(s) > 1$ and $p \nmid N$, we have the following identity

$$(1 - p^{-2s})^{-1} \sum_{k=0}^{\infty} \frac{a_f(p^k) a_g(p^k)}{p^{ks}}$$

$$= \left(1 - \epsilon_p \delta_p p^{-s}\right)^{-1} \left(1 - \epsilon_p \bar{\delta}_p p^{-s}\right)^{-1} \left(1 - \bar{\epsilon}_p \delta_p p^{-s}\right)^{-1} \left(1 - \bar{\epsilon}_p \bar{\delta}_p p^{-s}\right)^{-1}.$$

Proof Let $p \nmid N$. We recall that the coefficients $a_f(n)$ and $a_g(n)$ satisfy the following:

$$a_f(p^k) = a_f(p) a_f(p^{k-1}) - a_f(p^{k-2}),$$

$$a_g(p^k) = a_g(p) a_g(p^{k-1}) - a_g(p^{k-2}).$$

Applying the above identities repeatedly yields

$$a_f(p^k) a_g(p^k) - a_f(p) a_f(p^{k-1}) a_g(p) a_g(p^{k-1}) + \left(a_f(p)^2 + a_g(p)^2 - 2\right) a_f(p^{k-2}) a_g(p^{k-2})$$

$$- a_f(p) a_f(p^{k-3}) a_g(p) a_g(p^{k-3}) + a_f(p^{k-4}) a_g(p^{k-4}) = 0. \tag{2.4}$$

Also by using the above relations between the coefficients $a_f(p),\ a_g(p)$ and the complex units $\epsilon_p,\ \delta_p$, we have

$$\left(1-\epsilon_p\delta_p p^{-s}\right)\left(1-\epsilon_p\bar{\delta}_p p^{-s}\right)\left(1-\bar{\epsilon}_p\delta_p p^{-s}\right)\left(1-\bar{\epsilon}_p\bar{\delta}_p p^{-s}\right)$$

$$= 1 - a_f(p)a_g(p)p^{-s} + \left(a_f(p)^2 + a_g(p)^2 - 2\right)p^{-2s} - a_f(p)a_g(p)p^{-3s} + p^{-4s}. \qquad (2.5)$$

Putting together (2.4) and (2.5), and following a tedious calculation, we arrive at

$$\left(1-\epsilon_p\delta_p p^{-s}\right)\left(1-\epsilon_p\bar{\delta}_p p^{-s}\right)\left(1-\bar{\epsilon}_p\delta_p p^{-s}\right)\left(1-\bar{\epsilon}_p\bar{\delta}_p p^{-s}\right)\sum_{k=0}^{\infty}\frac{a_f(p^k)a_g(p^k)}{p^{ks}}$$

$$= 1 - \frac{1}{p^{2s}},$$

which is equivalent to the statement of the lemma.
This completes the proof. □

Proposition 2.14 *The modified Rankin-Selberg convolution of the modular L-functions associated to two normalized eigenforms f and g has the following Euler product*

$$\begin{aligned} L(f\otimes g, s) &= \prod_{p\mid N}\left(1-a_f(p)a_g(p)p^{-s}\right)^{-1} \\ &\times \prod_{p\nmid N}\left(1-\epsilon_p\delta_p p^{-s}\right)^{-1}\left(1-\epsilon_p\bar{\delta}_p p^{-s}\right)^{-1}\left(1-\bar{\epsilon}_p\delta_p p^{-s}\right)^{-1}\left(1-\bar{\epsilon}_p\bar{\delta}_p p^{-s}\right)^{-1}. \end{aligned}$$

Proof First of all we recall that the coefficients of eigenforms are multiplicative and real (see Subsection 1.2.3). So we have

$$L(f\otimes g, s) = \zeta_N(2s)\prod_{\text{all primes}}\left(\sum_{k=0}^{\infty}\frac{a_f(p^k)a_g(p^k)}{p^{ks}}\right).$$

For $p \mid N$, since $a_f(p^k) = a_f(p)^k$ and $a_g(p^k) = a_g(p)^k$, we have

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{a_f(p^k)a_g(p^k)}{p^{ks}} &= \sum_{k=0}^{\infty}\frac{a_f(p)^k a_g(p)^k}{p^{ks}} \\ &= \left(1-a_f(p)a_g(p)p^{-s}\right)^{-1}. \end{aligned}$$

Using this and applying the previous lemma, we attain the result. □

2.3 Symmetric Square L-Function

The following lemma gives a new representation for the Rankin-Selberg convolution $L(f \times f, s)$ of the modular L-function associated to an eigenform f with itself.

Lemma 2.15 *Let f be an eigenform of weight k and level N. Then,*

$$L(f \times f, s) = \zeta_N(s) \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s}.$$

Proof On one hand, by the definition of

$$\zeta_N(s) = \sum_{\substack{n=1 \\ \text{g.c.d.}(n,N)=1}}^{\infty} \frac{1}{n^s}$$

we have

$$\zeta_N(s) \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s} = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

where $b(n) = \sum_{\substack{d \mid n \\ \text{g.c.d.}(d,N)=1}} a_f\left(\frac{n^2}{d^2}\right)$.

On the other hand, we have

$$(a_f(n))^2 = \sum_{\substack{d \mid n \\ \text{g.c.d.}(d,N)=1}} a_f\left(\frac{n^2}{d^2}\right)$$

(see [11], p. 163, Proposition 39). This completes the proof. □

Inspired by the previous lemma, we define the main object of this section.

Definition 2.16 *Let f be a normalized eigenform of weight k and level N. For $Re(s) > 1$, the symmetric square L-function $L(\mathrm{sym}^2 f, s)$ associated to f is defined by*

$$L(\mathrm{sym}^2 f, s) = \zeta_N(2s) \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s}.$$

Lemma 2.15 shows that the symmetric square L-function satisfies the following

$$\zeta_N(s)L(\mathrm{sym}^2 f, s) = \zeta_N(2s)L(f \times f, s) = L(f \otimes f, s).$$

This identity, together with Theorem 2.12, establishes a meromorphic continuation of $L(\mathrm{sym}^2 f, s)$ to $\mathbb{C}$. In 1975 Shimura [22] proved that the symmetric square L-function in fact has an analytic continuation to the whole complex plane.

The value of $L(\mathrm{sym}^2 f, s)$ at $s = 1$ is of special interest. By part *(iii)* of Theorem 2.12 and calculating the residue of $\zeta_N(s)$ at $s = 1$, one can deduce that,

$$L(\mathrm{sym}^2 f, 1) = \frac{\pi(4\pi)^k}{2N(k-1)!}\langle f, f\rangle. \tag{2.6}$$

The other important fact about the symmetric square L-function is that for square-free N, it satisfies a functional equation. Let

$$L_\infty(\mathrm{sym}^2, s) = \pi^{-\frac{3}{2}s}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right),$$

and let

$$\Lambda(\mathrm{sym}^2 f, s) = N^s L_\infty(\mathrm{sym}^2, s)L(\mathrm{sym}^2 f, s).$$

Then the symmetric square L-function satisfies the functional equation

$$\Lambda(\mathrm{sym}^2 f, s) = \Lambda(\mathrm{sym}^2 f, 1-s). \tag{2.7}$$

In fact this functional equation can be derived from the functional equation for $L(f \otimes f, s)$ and the functional equation (1.2). In part *(v)* of Theorem 2.12, we established the functional equation of $L(f \otimes f, s)$ for $N = 1$. For square-free N, the proof of the functional equation of $L(f \otimes g, s)$, and consequently the functional equation of $L(\mathrm{sym}^2 f, s)$, is due to Ogg (see [17], Theorem 6, p. 311).

Chapter 3

Non-Vanishing on the Line $Re(s) = 1$

3.1 Introduction

Let f and g be two eigenforms with respect to the family of the Hecke operators for $\Gamma_0(N)$ (see Subsection 1.2.3). Let $L_f(s) = \sum_{n=1}^{\infty} a_f(n)n^{-s}$ and $L_g(s) = \sum_{n=1}^{\infty} a_g(n)n^{-s}$ be the L-functions associated to f and g, respectively. Let

$$L(f \otimes g, s) = \zeta_N(2s) \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s}$$

be the modified Rankin-Selberg convolution of $L_f(s)$ and $L_g(s)$. Let $\langle f, g\rangle$ denote the Petersson inner product of f and g. In [17] (Theorem 4) the following is proved:

Theorem 3.1 (Ogg) *If $\langle f, g\rangle = 0$, then $L(f \otimes g, 1) \neq 0$.*

In this chapter we prove similar non-vanishing results for a certain family of Dirichlet series and their convolutions. The non-vanishing of many classical L-functions will be simple corollaries of our general theorems. Also as a consequence of our theorems, we will be able to extend Ogg's theorem to the line $Re(s) = 1$. We start by introducing an important family of Dirichlet series.

3.2 A Class of Dirichlet Series

In 1989, Selberg [20] considered a certain class of Dirichlet series and announced a series of deep conjectures regarding the elements of that class.

Definition 3.2 *The Selberg class $\mathcal{S}$ is the family of functions $F(s)$ of a complex variable s satisfying the following properties:*

(i) (Dirichlet Series): for Re$(s) > 1$,

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$$

where $a_F(1) = 1$;

(ii) (Analytic Continuation): for some integer $m \geq 0$, $(s-1)^m F(s)$ extends to an entire function of finite order;

(iii) (Functional Equation): there are numbers $Q > 0,\ \alpha_i \geq 0,\ r_i \in \mathbb{C}$ with Re$(r_i) \geq 0$ so that

$$\Phi(s) = Q^s \prod_{i=1}^{d} \Gamma(\alpha_i s + r_i) F(s)$$

satisfies the functional equation

$$\Phi(s) = w\bar{\Phi}(1-s)$$

where w is a complex number with $|w| = 1$ and $\bar{\Phi}(s) = \overline{\Phi(\bar{s})}$;

(iv) (Euler Product): for Re$(s) > 1$,

$$F(s) = \prod_{p} \exp\left(\sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}\right)$$

where $b_F(p^k) = O(p^{k\theta})$ for some $\theta < 1/2$;

(v) (Ramanujan Hypothesis): for any fixed $\epsilon > 0$,

$$a_F(n) = O(n^{\epsilon})$$

where the implied constant may depend upon ϵ.

In our theorems, we only need to consider Dirichlet series that satisfy conditions similar to *(ii)*, *(iv)* and *(v)*. More precisely, we consider the following class.

Definition 3.3 *The class $\bar{\mathcal{S}}$ is the family of Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_F(n) n^{-s}$ (Re$(s) > 1$) satisfying the following properties:*

(a) For $Re(s) > 1$, we have

$$F(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}\right);$$

(b) For any fixed $\epsilon > 0$,

$$a_F(n) = O(n^\epsilon)$$

where the implied constant may depend upon ϵ.

(c) $F(s)$ has an analytic continuation to the line $Re(s) = 1$, except for a possible pole at point $s = 1$.

For $F \in \bar{\mathcal{S}}$, we write

$$\bar{F}(s) = \overline{F(\bar{s})} = \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^s} = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{\overline{b_F(p^k)}}{p^{ks}}\right).$$

Note that if F is analytic in a region A, then $\bar{F}$ is analytic in the region $\bar{A} = \{\bar{s} : s \in A\}$. So, if A be a symmetric region with respect to the real axis (the case that we are dealing with in this chapter), then the analyticity domain of F and $\bar{F}$ are the same. Also note that if F be the analytic continuation of f in a symmetric region, then $\bar{F}$ will be the analytic continuation of $\bar{f}$.

We continue by defining a convolution operation on $\bar{\mathcal{S}}$.

Definition 3.4 *For $F, G \in \bar{\mathcal{S}}$, the Euler product convolution of F and G is defined as*

$$(F \otimes G)(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{k b_F(p^k)\overline{b_G(p^k)}}{p^{ks}}\right).$$

Lemma 3.5 *For F, G in $\bar{\mathcal{S}}$, $(F \otimes G)(s)$ is convergent for $Re(s) > 1$.*

Proof First we show that $|a_F(n)| \leq c(\epsilon)n^\epsilon$ implies

$$|b_F(p^k)| \leq \frac{c(\epsilon)(2^k - 1)p^{k\epsilon}}{k}. \tag{3.1}$$

By taking logarithmic derivative of both sides of

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}\right)$$

we come to

$$-\frac{F'(s)}{F(s)} = \sum_{p,k} \frac{kb_F(p^k)\log p}{p^{ks}}.$$

By cross multiplying and using the formula $F'(s) = -\sum_{n=1}^{\infty} \frac{a_F(n)\log n}{n^s}$, we get

$$a_F(n)\log n = \sum_{p^j|n} jb_F(p^j)a_F(\frac{n}{p^j})\log p.$$

In particular for $n = p^k$ we have

$$kb_F(p^k)\log p = ka_F(p^k)\log p - \sum_{j=1}^{k-1} jb_F(p^j)a_F(p^{k-j})\log p$$

or

$$kb_F(p^k) = ka_F(p^k) - \sum_{j=1}^{k-1} jb_F(p^j)a_F(p^{k-j}).$$

We prove (3.1) by induction on k. For $k = 1$ we have $a_F(p) = b_F(p)$ and (3.1) is clear. Now suppose that (3.1) holds for all $j \le k-1$. We have

$$\begin{aligned} k|b_F(p^k)| &\le c(\epsilon)kp^{k\epsilon} + \sum_{j=1}^{k-1} j|b_F(p^j)|c(\epsilon)p^{(k-j)\epsilon} \\ &\le c(\epsilon)p^{k\epsilon}\left\{k + \sum_{j=1}^{k-1}(2^j - 1)\right\} \\ &\le c(\epsilon)p^{k\epsilon}(2^k - 1). \end{aligned}$$

This proves (3.1) for any p and k.

Now we prove the lemma. Suppose that $\sigma = Re(s) \ge 1 + 3\epsilon$. By the inequality $|e^z| \le e^{|z|}$ and by (3.1) we have

$$\begin{aligned} \left|\exp\left(\sum_{k=1}^{\infty} \frac{kb_F(p^k)\overline{b_G(p^k)}}{p^{ks}}\right)\right| &\le \exp\left(\sum_{k=1}^{\infty} \frac{k|b_F(p^k)||b_G(p^k)|}{p^{k\sigma}}\right) \\ &\ll \exp\left(\sum_{k=1}^{\infty} \frac{(2^k-1)^2 p^{2k\epsilon}/k}{p^{k\sigma}}\right). \end{aligned}$$

Replacing $2^k - 1$ by 2^k and using the expansion

$$-\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \tag{3.2}$$

valid for $|z| < 1$, we get

$$\begin{aligned}\left|\exp\left(\sum_{k=1}^{\infty}\frac{kb_F(p^k)\overline{b_G(p^k)}}{p^{ks}}\right)\right| &\ll \exp\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{4}{p^{\sigma-2\epsilon}}\right)^k\right)\\ &= \left(1-\frac{4}{p^{\sigma-2\epsilon}}\right)^{-1}.\end{aligned}$$

Since $\sigma - 2\epsilon \geq 1 + \epsilon > 1$, the series $\sum_p \frac{4}{p^{\sigma-2\epsilon}}$ is convergent. Therefore the product $\prod_p \left(1 - \frac{4}{p^{\sigma-2\epsilon}}\right)$ is also convergent, and nonzero (see [1], p. 191). This implies that

$$|(F \otimes G)(s)| \ll \prod_p \left(1 - \frac{4}{p^{\sigma-2\epsilon}}\right)^{-1} < \infty.$$

The proof is complete. □

In the following lemma we will show that $\zeta(s),\ L_\chi(s),\ L_f(s),\ L_{f,\chi}(s)$ and $L(f\otimes g, s)$ (for normalized eigenforms f and g) are all in $\bar{\mathcal{S}}$. Note that the conditions *(b)* and *(c)* in the definition of $\bar{\mathcal{S}}$ are clearly satisfied for these Dirichlet series, so we only need to check the condition *(a)* for them. Furthermore, we will establish the basic properties of the Euler product convolution.

Lemma 3.6 *(i) $\zeta(s)$ is in $\bar{\mathcal{S}}$, and for any F in $\bar{\mathcal{S}}$, we have*

$$(F \otimes \zeta)(s) = F(s).$$

(ii) For F in $\bar{\mathcal{S}}$, we have

$$(\zeta \otimes F)(s) = \bar{F}(s).$$

*(iii) If χ is a Dirichlet character (*mod q*), then $L_\chi(s)$ is in $\bar{\mathcal{S}}$, and*

$$(L_\chi \otimes L_\chi)(s) = \zeta_q(s).$$

(iv) Let f be a normalized eigenform in $S_k(N)$. Then $L_f(s)$ is in $\bar{\mathcal{S}}$, and

$$(L_f \otimes L_\chi)(s) = L_{f,\overline{\chi}}(s).$$

(v) For any two normalized eigenforms f and g in $S_k(N)$, $(L_f \otimes L_g)(s)$ is in $\bar{\mathcal{S}}$, and

$$(L_f \otimes L_g)(s) = L(f \otimes g, s).$$

Proof *(i)* By using the expansion (3.2), we get

$$\begin{aligned}\zeta(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \prod_p \exp\left(-\log\left(1 - \frac{1}{p^s}\right)\right) \\ &= \prod_p \exp\left(\frac{1}{p^s} + \frac{1/2}{p^{2s}} + \frac{1/3}{p^{3s}} + \cdots\right).\end{aligned}$$

This shows that $\zeta(s)$ is in $\bar{\mathcal{S}}$. For $F \in \bar{\mathcal{S}}$, we have

$$\begin{aligned}(F \otimes \zeta)(s) &= \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{k b_F(p^k)\frac{1}{k}}{p^{ks}}\right) \\ &= \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}\right) \\ &= F(s).\end{aligned}$$

(ii) Similarly we have

$$\begin{aligned}(\zeta \otimes F)(s) &= \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{k\frac{1}{k}\overline{b_F(p^k)}}{p^{ks}}\right) \\ &= \bar{F}(s).\end{aligned}$$

(iii) Similar to *(i)*, by using (3.2) and for $Re(s) > 1$, we have

$$\begin{aligned}L_\chi(s) &= \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1} \\ &= \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{\chi(p)^k/k}{p^{ks}}\right).\end{aligned}$$

This proves that $L_\chi(s)$ is in $\bar{\mathcal{S}}$. Moreover,

$$\begin{aligned}(L_\chi \otimes L_\chi)(s) &= \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{\chi(p^k)\overline{\chi}(p^k)/k}{p^{ks}}\right) \\ &= \prod_{p \nmid q} \exp\left(\sum_{k=1}^{\infty} \frac{1/k}{p^{ks}}\right) \\ &= \zeta_q(s).\end{aligned}$$

(iv) L_f is in $\bar{\mathcal{S}}$, since by Corollary 1.5, we have

$$\begin{aligned} L_f(s) &= \prod_{p|N}\left(1-a_f(p)p^{-s}\right)^{-1}\prod_{p\nmid N}\left(1-\epsilon_p p^{-s}\right)^{-1}\left(1-\bar{\epsilon}_p p^{-s}\right)^{-1} \\ &= \prod_{p|N}\exp\left(\sum_{k=1}^{\infty}\frac{a_f(p)^k/k}{p^{ks}}\right)\prod_{p\nmid N}\exp\left(\sum_{k=1}^{\infty}\frac{\left(\epsilon_p^k+\bar{\epsilon}_p^k\right)/k}{p^{ks}}\right). \end{aligned}$$

We have

$$(L_f\otimes L_\chi)(s)=\prod_{p|N}\exp\left(\sum_{k=1}^{\infty}\frac{a_f(p)^k\overline{\chi}(p)^k/k}{p^{ks}}\right)\prod_{p\nmid N}\exp\left(\sum_{k=1}^{\infty}\frac{(\epsilon_p^k+\bar{\epsilon}_p^k)\overline{\chi}(p)^k/k}{p^{ks}}\right).$$

Also note that since $a_f(n)$ and $\chi(n)$ are multiplicative,

$$L_{f,\overline{\chi}}(s)=\prod_p\left(\sum_{l=0}^{\infty}\frac{a_f(p^l)\overline{\chi}(p)^l}{p^{ls}}\right).$$

If $p \mid N$, the corresponding p-factor for $(L_f\otimes L_\chi)(s)$ is

$$\exp\left(\sum_{k=1}^{\infty}\frac{a_f(p)^k\overline{\chi}(p)^k/k}{p^{ks}}\right)=\left(1-a_f(p)\overline{\chi}(p)p^{-s}\right)^{-1}$$

and the corresponding p-factor for $L_{f,\overline{\chi}}(s)$ is also

$$\sum_{l=0}^{\infty}\frac{a_f(p)^l\overline{\chi}(p)^l}{p^{ls}}=\left(1-a_f(p)\overline{\chi}(p)p^{-s}\right)^{-1}.$$

Now suppose that $p \nmid N$. To prove that the corresponding p-factors in the Euler products of $(L_f\otimes L_\chi)(s)$ and $L_{f,\overline{\chi}}(s)$ are equal, we need to prove that

$$\exp\left(\sum_{k=1}^{\infty}\frac{(\epsilon_p^k+\bar{\epsilon}_p^k)\overline{\chi}(p)^k/k}{p^{ks}}\right)=\sum_{k=0}^{\infty}\frac{a_f(p^k)\overline{\chi}(p)^k}{p^{ks}}.$$

These two quantities are equal if and only if the following equality holds

$$(1-\epsilon_p\overline{\chi}(p)p^{-s})^{-1}(1-\bar{\epsilon}_p\overline{\chi}(p)p^{-s})^{-1}=\sum_{k=0}^{\infty}\frac{a_f(p^k)\overline{\chi}(p)^k}{p^{ks}}$$

or

$$\left(1 - a_f(p)\overline{\chi}(p)p^{-s} + \overline{\chi}(p)^2 p^{-2s}\right) \sum_{k=0}^{\infty} \frac{a_f(p^k)\overline{\chi}(p)^k}{p^{ks}} = 1.$$

After expanding, the left-hand side becomes

$$\sum_{k=0}^{\infty} \frac{a_f(p^k)\overline{\chi}(p)^k}{p^{ks}} - \sum_{k=0}^{\infty} \frac{a_f(p)a_f(p^k)\overline{\chi}(p)^{k+1}}{p^{(k+1)s}} + \sum_{k=0}^{\infty} \frac{a_f(p^k)\overline{\chi}(p)^{k+2}}{p^{(k+2)s}}$$

or

$$1 + \frac{a_f(p)\overline{\chi}(p)}{p^s} - \frac{a_f(p)\overline{\chi}(p)}{p^s} + \sum_{k=0}^{\infty} \frac{(a_f(p^{k+2}) - a_f(p)a_f(p^{k+1}) + a_f(p^k))\overline{\chi}(p)^{k+2}}{p^{(k+2)s}},$$

which is clearly equal to 1. Therefore $(L_f \otimes L_\chi)(s) = L_{f,\overline{\chi}}(s)$.

(v) On one hand, from part *(iv)*, we have

$$L_f(s) = \prod_{p|N} \exp\left(\sum_{k=1}^{\infty} \frac{a_f(p)^k/k}{p^{ks}}\right) \prod_{p\nmid N} \exp\left(\sum_{k=1}^{\infty} \frac{(\epsilon_p^k + \bar{\epsilon}_p^k)/k}{p^{ks}}\right)$$

and

$$L_g(s) = \prod_{p|N} \exp\left(\sum_{k=1}^{\infty} \frac{a_g(p)^k/k}{p^{ks}}\right) \prod_{p\nmid N} \exp\left(\sum_{k=1}^{\infty} \frac{(\delta_p^k + \bar{\delta}_p^k)/k}{p^{ks}}\right).$$

So, we have

$$\begin{aligned} (L_f \otimes L_g)(s) &= \prod_{p|N} \exp\left(\sum_{k=1}^{\infty} \frac{a_f(p)^k a_g(p)^k/k}{p^{ks}}\right) \prod_{p\nmid N} \exp\left(\sum_{k=1}^{\infty} \frac{(\epsilon_p^k + \bar{\epsilon}_p^k)(\delta_p^k + \bar{\delta}_p^k)/k}{p^{ks}}\right) \\ &= \prod_{p|N} (1 - a_f(p)a_g(p)p^{-s})^{-1} \\ &\times \prod_{p\nmid N} (1 - \epsilon_p\delta_p p^{-s})^{-1}(1 - \epsilon_p\bar{\delta}_p p^{-s})^{-1}(1 - \bar{\epsilon}_p\delta_p p^{-s})^{-1}(1 - \bar{\epsilon}_p\bar{\delta}_p p^{-s})^{-1}. \end{aligned}$$

This shows that $(L_f \otimes L_g)(s)$ is in $\bar{\mathcal{S}}$.

On the other hand, from Proposition 2.14 we know that

$$\begin{aligned} L(f \otimes g, s) &= \prod_{p|N} \left(1 - a_f(p)a_g(p)p^{-s}\right)^{-1} \\ &\times \prod_{p\nmid N} (1 - \epsilon_p\delta_p p^{-s})^{-1}(1 - \epsilon_p\bar{\delta}_p p^{-1})^{-1}(1 - \bar{\epsilon}_p\delta_p p^{-s})^{-1}(1 - \bar{\epsilon}_p\bar{\delta}_p p^{-s})^{-1}. \end{aligned}$$

Thus, $(L_f \otimes L_g)(s) = L(f \otimes g, s)$.

This completes the proof. $\square$

3.3 Mertens's Method

As we mentioned in Section 1.1, the main step in the proof of the Prime Number Theorem is establishing the non-vanishing of the Riemann zeta-function on the line $Re(s) = 1$. This fact was proved by Hadamard and de la Vallée Poussin in 1896. In 1898 Mertens gave a simpler proof for this fact. Mertens's proof depends upon the choice of a suitable trigonometric inequality. This line of proof is adaptable for establishing the non-vanishing of various L-functions. In [18], Rankin used this method to prove the non-vanishing of $L_f(s)$ on the line $Re(s) = 1$, $s \neq 1$, where f is an eigenform for $\Gamma_0(N)$. The proof of the following lemma, due to K. Murty [16], which is similar to the Mertens's proof, depends on a certain trigonometric inequality.

Lemma 3.7 *Let $f(s)$ be a complex function satisfying the following:*
(i) $f(s)$ is analytic in $Re(s) > 1$ and non-zero there;
(ii) $\log f(s)$ can be written as a Dirichlet series

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

with $b_n \geq 0$ for $Re(s) > 1$;
(iii) On the line $Re(s) = 1$, $f(s)$ is analytic except for a pole of order $e \geq 0$ at $s = 1$.
Then, if $f(s)$ has a zero on the line $Re(s) = 1$, the order of that zero is bounded by $\dfrac{e}{2}$.

Proof Suppose f has a zero at point $1 + it_0$ ($t_0 \neq 0$) of order $k > \frac{e}{2}$. Then $e \leq 2k - 1$. Now consider the function

$$g(s) = f(s)^{2k+1} f(s + it_0)^{4k} f(s + 2it_0)^{4k-2} \cdots f(s + 2kit_0)^2.$$

$g(s)$ is analytic for $Re(s) > 1$ and vanishes at $s = 1$ as

$$(4k)k - (2k + 1)e \geq 4k^2 - (2k + 1)(2k - 1) = 1.$$

Note that for $Re(s) > 1$,

$$\log g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \left(2k + 1 + 2\sum_{j=1}^{2k} (2k+1-j) n^{-ijt_0} \right).$$

Now let $\theta_n = t_0 \log n$. Then for $s = \sigma > 1$,

$$\begin{aligned} \log |g(\sigma)| &= Re(\log g(\sigma)) \\ &= \sum_{n=1}^{\infty} \frac{b_n}{n^\sigma} \left(2k + 1 + 2\sum_{j=1}^{2k} (2k+1-j) \cos j\theta_n \right). \end{aligned}$$

Applying the trigonometric identity

$$2k + 1 + 2\sum_{j=1}^{2k} (2k+1-j) \cos j\theta = \left(1 + 2\sum_{j=1}^{k} \cos j\theta \right)^2$$

in the previous equality yields

$$\log |g(\sigma)| = \sum_{n=1}^{\infty} \frac{b_n}{n^\sigma} \left(1 + 2\sum_{j=1}^{k} \cos j\theta_n \right)^2 \geq 0.$$

Hence, $\log |g(\sigma)| \geq 0$ for $\sigma > 1$, i.e., $|g(\sigma)| \geq 1$. So

$$0 = |g(1)| = \lim_{\sigma \to 1^+} |g(\sigma)| \geq 1,$$

which is a contradiction.
The proof is complete. □

Corollary 3.8 *(i)* $\zeta(1+it) \neq 0$ *for* $t \neq 0$.
(ii) For any Dirichlet character χ (mod q), $L_\chi(1+it) \neq 0$ *for* $t \neq 0$.
(iii) If χ *is complex (i.e.,* $\chi \neq \overline{\chi}$*),* $L_\chi(1) \neq 0$.

Proof *(i)* We know that $\zeta(s)$ has an analytic continuation to the whole complex plane, except for a simple pole at $s = 1$ (see Section 1.1) and is nonzero for $Re(s) > 1$. Also by (3.2), we have

$$\log \zeta(s) = \sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{ks}}.$$

Therefore, by Lemma 3.7, we are done.

(ii) Consider the following product

$$f(s) = \prod_{\chi} L_{\chi}(s).$$

$f(s)$ has an analytic continuation to the whole complex plane, except for a simple pole at $s = 1$ (which comes from $L_{\chi_0}(s)$), and is nonzero for $Re(s) > 1$. Now by applying the *orthogonality relations* for characters (mod q), i.e.,

$$\frac{1}{\phi(q)} \sum_{\chi} \chi(a) = \begin{cases} 1 & \text{if } a \equiv 1 \bmod q \\ 0 & \text{otherwise} \end{cases}$$

(see [15], Exercise 2.2.9), and for $Re(s) > 1$, we have

$$\begin{aligned} \log f(s) &= \sum_{\chi} \log L_{\chi}(s) \\ &= \phi(q) \sum_{\substack{p,k \\ p^k \equiv 1 \bmod q}} \frac{1}{kp^{ks}}. \end{aligned}$$

This is a Dirichlet series with non-negative coefficients. Thus, by Lemma 3.7, $f(1+it) \neq 0$ for $t \neq 0$. This proves *(ii)*.

(iii) Suppose that for a complex character χ_1, $L_{\chi_1}(1) = 0$. Thus, $L_{\overline{\chi}_1}(1) = 0$. Since $\chi_1 \neq \overline{\chi}_1$, and since all factors $L_{\chi}(s)$ of $f(s)$ are analytic at $s = 1$ except $L_{\chi_0}(s)$, we conclude that $f(s)$ is in fact analytic at $s = 1$ and $f(1) = 0$. This violates the statement of the Lemma 3.7.

The proof is complete. □

Definition 3.9 *For $F \in \bar{\mathcal{S}}$ and $\sigma_0 \leq 1$, we say F is $\otimes$-simple in $Re(s) > \sigma_0$ (resp. $Re(s) \geq \sigma_0$), if $F \otimes F$ has an analytic continuation to $Re(s) > \sigma_0$ (resp. $Re(s) \geq \sigma_0$), except for a possible simple pole at $s = 1$.*

The following theorem is the main result of this section.

Theorem 3.10 *Let F, $G \in \bar{\mathcal{S}}$ be $\otimes$-simple in $Re(s) \geq 1$ and $t \neq 0$. Then*

(i) $(F \otimes F)(1 + it) \neq 0$.

(ii) If $F = \bar{F}$, $G = \bar{G}$, and if $F \otimes G$ has an analytic continuation to the line $Re(s) = 1$, then $(F \otimes G)(1 + it) \neq 0$.

Proof *(i)* Let $f(s) = (F \otimes F)(s)$. We have

$$\log f(s) = \sum_p \sum_{k=1}^{\infty} \frac{k|b_F(p^k)|^2}{p^{ks}} = \sum_{n=1}^{\infty} \frac{c(n)}{n^s},$$

with $c(n) \geq 0$. So, $f(s)$ satisfies the conditions of Lemma 3.7 with $e = 1$. Therefore, the order of the vanishing of $f(s)$ at point $1+it$ is $\leq \frac{1}{2}$. This means that $(F\otimes F)(1+it) \neq 0$.

(ii) Let

$$f(s) = (F \otimes F)(s)((F \otimes G)(s))^2(G \otimes G)(s).$$

Since for $t \neq 0$, all the factors of $f(s)$ have finite values at point $1 + it$, in order to prove that $(F \otimes G)(1 + it) \neq 0$, it suffices to show that $f(1 + it) \neq 0$. Note that

$$\begin{aligned}
\log f(s) &= \sum_p \sum_{k=1}^{\infty} \frac{kb_F(p^k)^2}{p^{ks}} + 2\sum_p \sum_{k=1}^{\infty} \frac{kb_F(p^k)b_G(p^k)}{p^{ks}} + \sum_p \sum_{k=1}^{\infty} \frac{kb_G(p^k)^2}{p^{ks}} \\
&= \sum_p \sum_{k=1}^{\infty} \frac{k\left(b_F(p^k) + b_G(p^k)\right)^2}{p^{ks}} \\
&= \sum_{n=1}^{\infty} \frac{c(n)}{n^s}
\end{aligned}$$

with $c(n) \geq 0$. So, $f(s)$ satisfies the conditions of Lemma 3.7 with $e \leq 2$, and therefore, the order of the vanishing of $f(s)$ at point $1+it$ is ≤ 1. Now suppose that $f(1+it) = 0$. Thus,

$$(F \otimes F)(1 + it)\left((F \otimes G)(1 + it)\right)^2 (G \otimes G)(1 + it) = 0.$$

Since by part *(i)*, $(F \otimes F)(1 + it) \neq 0$ and $(G \otimes G)(1 + it) \neq 0$, it follows that $(F \otimes G)(1 + it) = 0$. This is a contradiction; otherwise, the order of the vanishing of $f(s)$ at point $1 + it$ should be 2.
This completes the proof. □

Corollary 3.11 *If* $F = \bar{F} \in \bar{\mathcal{S}}$ *is analytic and* $\otimes$*-simple in* $Re(s) \geq 1$*, then* $F(1+it) \neq 0$ *for* $t \neq 0$.

Proof This is a simple consequence of part *(ii)* of the previous theorem with $G(s) = \zeta(s)$. $\square$

Corollary 3.12 *Let $f \in S_k(N)$ be an eigenform for $\Gamma_0(N)$, let χ be a real character* $(\bmod\ q)$ *and let $t \neq 0$. Then*

(i) $L_\chi(1+it) \neq 0$ *and* $L_f(1+it) \neq 0$.

(ii) $L_{f,\chi}(1+it) \neq 0$.

(iii) $L(f \otimes f, 1+it) \neq 0$ *and* $L(\mathrm{sym}^2 f, 1+it) \neq 0$.

(iv) Suppose $g \in S_k(N)$ is also an eigenform for $\Gamma_0(N)$. If $\langle f, g\rangle = 0$, then $L(f \otimes g, 1+it) \neq 0$.

Proof Without loss of generality, we can assume that f is normalized.

(i) Since $L_\chi(s)$ $(\chi \neq \chi_0)$ and $L_f(s)$ are analytic on the line $Re(s) = 1$, by the previous corollary, we have the desired result. Note that if $\chi = \chi_0$, since $(L_{\chi_0} \otimes L_{\chi_0})(s) = \prod_{p|q}(1-p^{-s})\zeta(s)$, the result is clear by part *(i)* of Theorem 3.10 .

(ii) We know that $L_f(s)$ and $L_\chi(s)$ are in $\bar{\mathcal{S}}$. Since $L_{f,\chi}(s)$ is the L-function associated to a cusp form (see Subsection 1.2.2), so $L_{f,\chi}(s)$ is analytic on the line $Re(s) = 1$. Therefore, by part *(iv)* of Lemma 3.6 and part *(ii)* of Theorem 3.10,

$$L_{f,\chi}(1+it) = (L_f \otimes L_\chi)(1+it) \neq 0.$$

(iii) Since $L_f(s) \in \bar{\mathcal{S}}$ has all the necessary conditions, by part *(i)* of Theorem 3.10, we have

$$L(f \otimes f, 1+it) = (L_f \otimes L_f)(1+it) \neq 0.$$

Also, since $L(f\otimes f, s) = \zeta_N(2s)L(\mathrm{sym}^2 f, s)$, we have the desired result for $L(\mathrm{sym}^2 f, s)$.

(iv) Recall that the coefficients of eigenforms are real (see Subsection 1.2.3). If $\langle f, g\rangle = 0$, by Theorem 2.12, we know that $L(f \otimes g, s)$ is actually an entire function. Therefore, by part *(ii)* of Theorem 3.10, $L(f \otimes g, 1+it) \neq 0$.
This completes the proof. $\square$

3.4 Landau's Theorem

In the previous section we proved a general non-vanishing result on the line $Re(s) = 1$ when $s \neq 1$. In this section, we consider the non-vanishing problem for $s = 1$. To do this, our basic ingredient is the following lemma.

Lemma 3.13 (Landau) *A Dirichlet series with non-negative coefficients has a singularity at its abscissa of convergence.*

Proof Let σ_0 be the abscissa of convergence for the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

and suppose that $a(n) \geq 0$. If f is not singular at σ_0, then there is a power series representation for f

$$f(s) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\sigma_1)}{k!}(s - \sigma_1)^k$$

for $|s - \sigma_1| < \epsilon$, where $\sigma_1 - \epsilon < \sigma_0 < \sigma_1$. Now for $\sigma_1 - \epsilon < \sigma < \sigma_1$ we have

$$\begin{aligned} f(\sigma) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(\sigma_1)}{k!}(\sigma - \sigma_1)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k f^{(k)}(\sigma_1)}{k!}(\sigma_1 - \sigma)^k. \end{aligned}$$

Since $\sigma_1 > \sigma$, by the well-known formula for the successive derivatives of a convergent Dirichlet series, we can write

$$\begin{aligned} f(\sigma) &= \sum_{k=0}^{\infty} \left(\frac{(\sigma_1 - \sigma)^k}{k!} \sum_{n=1}^{\infty} \frac{a(n)(\log n)^k}{n^{\sigma_1}} \right) \\ &= \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{a(n)\left((\sigma_1 - \sigma)\log n\right)^k}{k! n^{\sigma_1}}. \end{aligned}$$

However, since the last double series has non-negative terms, we are allowed to interchange the order of summation to get

$$\begin{aligned} f(\sigma) &= \sum_{n=1}^{\infty} \left(\frac{a(n)}{n^{\sigma_1}} \sum_{k=0}^{\infty} \frac{\left((\sigma_1 - \sigma)\log n\right)^k}{k!} \right) \\ &= \sum_{n=1}^{\infty} \frac{a(n)}{n^{\sigma_1}} n^{\sigma_1 - \sigma} \\ &= \sum_{n=1}^{\infty} \frac{a(n)}{n^{\sigma}}, \end{aligned}$$

which is a contradiction with our assumption that σ_0 is the abscissa of convergence of the series.
This completes the proof. $\square$

Corollary 3.14 *Let $f(s)$ be a complex function that satisfies the following:*

(i) $f(s)$ is analytic on the half-plane $Re(s) > \sigma_0$;

(ii) $\log f(s)$ has a representation in terms of a Dirichlet series with non-negative coefficients on the half-plane $Re(s) > \sigma_1$ ($\sigma_1 > \sigma_0$).
Then $f(s) \neq 0$ for $Re(s) > \sigma_0$.

Proof Let σ_2 be the largest real zero of f ($\sigma_0 < \sigma_2 \leq \sigma_1$). Since $\log f(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$ for $Re(s) > \sigma_1$ ($c(n) \geq 0$), and since $\log f(s)$ is analytic in a neighbourhood of the segment $\sigma_2 < \sigma \leq \sigma_1$, then by the previous lemma, we have

$$\log f(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

for $Re(s) > \sigma_2$. Thus,

$$\begin{aligned} \log |f(\sigma)| &= Re(\log f(\sigma)) \\ &= \log f(\sigma) \\ &= \sum_{n=1}^{\infty} \frac{c(n)}{n^\sigma} \geq 0 \end{aligned}$$

for $\sigma > \sigma_2$. Therefore, $|f(\sigma)| \geq 1$ for $\sigma > \sigma_2$. This contradicts the assumption $f(\sigma_2) = 0$, and therefore f has no real zero $\sigma > \sigma_0$. So $\log f(s)$ is analytic on the interval $(\sigma_0,\ \sigma_1]$, and Lemma 3.13 in fact shows that $\log f(s)$ exists and is analytic for $Re(s) > \sigma_0$. This means that $f(s)$ is non-zero for $Re(s) > \sigma_0$.
The proof is complete. $\square$

Here, we prove the main result of this section.

Theorem 3.15 *Let $\sigma_0 < 1$, and assume the following:*

(i) F and G (as members of $\bar{\mathcal{S}}$) are $\otimes$-simple in $Re(s) > \sigma_0$;

(ii) $F \otimes G$ has an analytic continuation to the half-plane $Re(s) > \sigma_0$;

(iii) At least one of $F \otimes F$, $G \otimes G$, or $F \otimes G$ has zeros in the half-plane $Re(s) > \sigma_0$.
Then $(F \otimes G)(1) \neq 0$.

Proof Suppose that $(F \otimes G)(1) = 0$, and let

$$f(s) = (F \otimes F)(s)\ (F \otimes G)(s)\ (\bar{F} \otimes \bar{G})(s)\ (G \otimes G)(s).$$

First of all note that $\bar{F} \otimes \bar{G}$ is analytic for $Re(s) > \sigma_0$. Since $(F \otimes G)(1) = 0$, then $(\bar{F} \otimes \bar{G})(1) = 0$, and since $s = 1$ is a pole of order ≤ 1 for both $F \otimes F$ and $G \otimes G$, we conclude that $f(s)$ is analytic at point $s = 1$, and therefore, analytic for $Re(s) > \sigma_0$. Now note that for $Re(s) > 1$,

$$\begin{aligned}
\log f(s) &= \sum_p \sum_{k=1}^{\infty} \frac{k|b_F(p^k)|^2}{p^{ks}} + \sum_p \sum_{k=1}^{\infty} \frac{k b_F(p^k)\overline{b_G(p^k)}}{p^{ks}} \\
&+ \sum_p \sum_{k=1}^{\infty} \frac{k\overline{b_F(p^k)} b_G(p^k)}{p^{ks}} + \sum_p \sum_{k=1}^{\infty} \frac{k|b_G(p^k)|^2}{p^{ks}} \\
&= \sum_p \sum_{k=1}^{\infty} \frac{k|b_F(p^k) + b_G(p^k)|^2}{p^{ks}} \\
&= \sum_{n=1}^{\infty} \frac{c(n)}{n^s}
\end{aligned}$$

where $c(n) \geq 0$. So, $f(s)$ satisfies the conditions of the Corollary 3.14 with $\sigma_1 = 1$, and therefore, $f(s) \neq 0$ for $Re(s) > \sigma_0$. This contradicts our assumption in *(iii)*.
The proof is complete. □

Corollary 3.16 *Let $F \in \bar{\mathcal{S}}$ be analytic and $\otimes$-simple in $Re(s) \geq \frac{1}{2}$, then $F(1) \neq 0$.*

Proof Let $G(s) = \zeta(s)$. Note that $F \otimes G = F$ and $\bar{G} = G$. Also notice that $\zeta(s)$ has zeros in the half-plane $Re(s) \geq 1/2$. Thus, all the conditions of the Theorem 3.15 are met with $\sigma_0 \leq \frac{1}{2}$. Therefore, $F(1) = (F \otimes G)(1) \neq 0$. □

Corollary 3.17 *Let $f, g \in S_k(N)$ be eigenforms for $\Gamma_0(N)$, and let χ be a Dirichlet character* (mod q)*. Then*

(i) If $\chi \neq \chi_0$, then $L_\chi(1) \neq 0$.
(ii) $L_f(1) \neq 0$.
(iii) $L_{f,\chi}(1) \neq 0$.
(iv) If $\langle f, g \rangle = 0$, then $L(f \otimes g, 1) \neq 0$.

Proof *(i)* By part *(iii)* of Lemma 3.6, we have

$$(L_\chi \otimes L_\chi)(s) = \zeta_q(s).$$

This shows that $L_\chi(s)$ satisfies the conditions in Corollary 3.16, and therefore $L_\chi(1) \neq 0$.

(ii) By part *(v)* of Lemma 3.6, we have

$$(L_f \otimes L_f)(s) = L(f \otimes f, s).$$

This and Theorem 2.12 imply that $L_f(s)$ is $\otimes$-simple in $Re(s) \geq 1/2$. So by Corollary 3.16 we are done.

(iii) Note that $(L_\chi \otimes L_\chi)(s)$ and $(L_f \otimes L_f)(s)$ can be extended analytically to the whole complex plane, except for a simple pole at $s = 1$. Also by part *(iv)* of Lemma 3.6, we have

$$L_{f,\chi}(s) = (L_f \otimes L_\chi)(s).$$

We know that $L_{f,\chi} \in S_k(q^2N)$ (see Subsection 1.2.2). So, $(L_f \otimes L_\chi)(s)$ has an analytic continuation to the whole complex plane. Also note that $(L_\chi \otimes L_\chi)(s) = \zeta_q(s)$ has in fact infinitely many zeros (see [7], p. 97). So all the conditions of Theorem 3.15 are met and therefore, $L_{f,\chi}(1) = (L_f \otimes L_\chi)(1) \neq 0$.

(iv) Similar to the proof of part *(iii)*, we can show that the conditions *(i)* and *(ii)* of Theorem 3.15 are satisfied. The result will be obtained if we only show that $L(f \otimes g, s)$ has a zero in the complex plane. By (2.3), if $\langle f, g \rangle = 0$, then

$$\Phi(s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)L(f \otimes g, s)$$

is analytic at $s = 0$. Since $\Gamma(s)$ has a pole at $s = 0$, then $L(f \otimes g, 0) = 0$.
This completes the proof. □

3.5 Ingham's Proof

In the previous section we studied the non-vanishing of certain Dirichlet series at point $s = 1$. Comparing the different methods we have applied in Sections 3.3 and 3.4, one realizes that the non-vanishing problem at point $s = 1$ has a distinct nature, and it seems that Landau's Theorem cannot be applied to prove non-vanishing results for other points on the line $Re(s) = 1$. However, in this section, we will show that for Dirichlet series with completely multiplicative coefficients, one can apply the technique employed in the previous section to prove a non-vanishing result on the line $Re(s) = 1$. Our result is a generalization of Ingham's proof of the non-vanishing of the Riemann zeta-function on the line $Re(s) = 1$ [9]. To do this, we start with the following definitions.

Definition 3.18 *Let $F, G \in \bar{\mathcal{S}}$. Then the L-convolution*[1] *of F and G is defined by*

$$L(F \otimes G, s) = \sum_{n=1}^{\infty} \frac{a_F(n)\overline{a_G(n)}}{n^s}.$$

An arithmetic function $f(n)$ is called multiplicative (resp. completely multiplicative) if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all m, n with g.c.d.$(m, n) = 1$ *(resp. for all m, n).*

Lemma 3.19 *For $F, G \in \bar{\mathcal{S}}$ with completely multiplicative coefficients,*

$$L(F \otimes G, s) = (F \otimes G)(s).$$

Proof We have

$$\begin{aligned} L(F \otimes G, s) &= \sum_{n=1}^{\infty} \frac{a_F(n)\overline{a_G(n)}}{n^s} \\ &= \prod_p \left(\sum_{k=0}^{\infty} \frac{a_F(p)^k \left(\overline{a_G(p)}\right)^k}{p^{ks}} \right) \\ &= \prod_p \left(1 - a_F(p)\overline{a_G(p)}p^{-s}\right)^{-1} \\ &= \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{a_F(p)^k \left(\overline{a_G(p)}\right)^k /k}{p^{ks}} \right) \\ &= (F \otimes G)(s). \end{aligned}$$

[1] We have chosen this name to distinguish this convolution from the Rankin-Selberg convolution. Note that $F(s)$ and $G(s)$ are not necessarily modular L-functions.

The last equality is true since

$$\begin{aligned} F(s) &= \prod_p \left(\sum_{k=0}^{\infty} \frac{a_F(p)^k}{p^{ks}} \right) \\ &= \prod_p \left(1 - a_F(p)p^{-s}\right)^{-1} \\ &= \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{a_F(p)^k/k}{p^{ks}} \right). \end{aligned}$$

and similarly

$$G(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{a_G(p)^k/k}{p^{ks}} \right).$$

The proof is complete. □

Definition 3.20 *If $f(n)$ is an arithmetic function, the formal L-series attached to $f(n)$ is defined by*

$$L(f, s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

If $g(n)$ is also an arithmetic function, the Dirichlet convolution of $f(n)$ and $g(n)$ is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}).$$

It can be shown that the Dirichlet convolution of two multiplicative arithmetic functions is multiplicative (see [3], Theorem 2.14, p. 35). The following identity of formal L-series, due to J. Borwein and Choi [6], will be fundamental in the proof of the main result of this section.

Lemma 3.21 *Let f_1, f_2, g_1, g_2 be completely multiplicative arithmetic functions. Then we have*

$$\sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n)(f_2 * g_2)(n)}{n^s} = \frac{L(f_1 f_2, s)L(g_1 g_2, s)L(f_1 g_2, s)L(f_2 g_1, s)}{L(f_1 f_2 g_1 g_2, 2s)}.$$

Proof We only need to prove that the corresponding p-factors in the Euler product of both sides are equal.

First suppose that $f_i \neq g_i$ $(i = 1, 2)$. Since both $f_1 * g_1$ and $f_2 * g_2$ are multiplicative, the left-hand side has an Euler product with the p-factor

$$L_{(p)}(s) = \sum_{k=0}^{\infty} \frac{(f_1 * g_1)(p^k)(f_2 * g_2)(p^k)}{p^{ks}}.$$

Also since all functions $f_1 f_2$, $g_1 g_2$, $f_1 g_2$, $f_2 g_1$ and $f_1 f_2 g_1 g_2$ are completely multiplicative, the following fraction is the corresponding p-factor in the Euler product of the right-hand side

$$R_{(p)}(s) = \frac{1 - (f_1 f_2 g_1 g_2)(p)p^{-2s}}{(1 - (f_1 f_2)(p)p^{-s})(1 - (g_1 g_2)(p)p^{-s})(1 - (f_1 g_2)(p)p^{-s})(1 - (f_2 g_1)(p)p^{-s})}.$$

Now by using the following elementary identity

$$\frac{\frac{ab}{1-abx} + \frac{cd}{1-cdx} - \frac{ad}{1-adx} - \frac{bc}{1-bcx}}{(a-c)(b-d)} = \frac{1 - abcdx^2}{(1-abx)(1-cdx)(1-adx)(1-bcx)}$$

and the fact that

$$(f_i * g_i)(p^k) = \sum_{d|p^k} f_i(d) g_i(\frac{p^k}{d}) = \frac{f_i(p)^{k+1} - g_i(p)^{k+1}}{f_i(p) - g_i(p)}$$

which comes from the complete multiplicativity, we have

$$\begin{aligned}
L_{(p)}(s) &= \sum_{k=0}^{\infty} (f_1 * g_1)(p^k)(f_2 * g_2)(p^k)p^{-ks} \\
&= \sum_{k=0}^{\infty} \frac{(f_1(p)^{k+1} - g_1(p)^{k+1})(f_2(p)^{k+1} - g_2(p)^{k+1})}{(f_1(p) - g_1(p))(f_2(p) - g_2(p))} p^{-ks} \\
&= \frac{\sum_{k=0}^{\infty} \{(f_1 f_2)(p)^{k+1} + (g_1 g_2)(p)^{k+1} - (f_1 g_2)(p)^{k+1} - (f_2 g_1)(p)^{k+1}\} p^{-ks}}{(f_1(p) - g_1(p))(f_2(p) - g_2(p))} \\
&= \frac{\frac{(f_1 f_2)(p)}{1-(f_1 f_2)(p)p^{-s}} + \frac{(g_1 g_2)(p)}{1-(g_1 g_2)(p)p^{-s}} - \frac{(f_1 g_2)(p)}{1-(f_1 g_2)(p)p^{-s}} - \frac{(f_2 g_1)(p)}{1-(f_2 g_1)(p)p^{-s}}}{(f_1(p) - g_1(p))(f_2(p) - g_2(p))} \\
&= \frac{1 - (f_1 f_2 g_1 g_2)(p)p^{-2s}}{(1 - (f_1 f_2)(p)p^{-s})(1 - (g_1 g_2)(p)p^{-s})(1 - (f_1 g_2)(p)p^{-s})(1 - (f_2 g_1)(p)p^{-s})} \\
&= R_{(p)}(s).
\end{aligned}$$

Now suppose that $f_1 = g_1$ and $f_2 \neq g_2$. The proof is similar to the previous case. The only difference is that this time one has to use the identity

$$\frac{\frac{b}{(1-abx)^2} - \frac{d}{(1-adx)^2}}{b-d} = \frac{1 - a^2bdx^2}{(1-abx)^2(1-adx)^2}.$$

Note that this also covers the case $f_1 \neq g_1$, $f_2 = g_2$.

Finally, if $f_1 = g_1$, $f_2 = g_2$, one needs to employ the identity

$$\frac{1+abx}{(1-abx)^3} = \frac{1-a^2b^2x^2}{(1-abx)^4}.$$

This completes the proof. □

We are ready to state and prove the main result of this section.

Theorem 3.22 *Let F, $G \in \bar{S}$ be two Dirichlet series with completely multiplicative coefficients. Also assume the following:*

(i) F and G are $\otimes$-simple in $Re(s) \geq \frac{1}{2}$;

(ii) $F \otimes G$ has an analytic continuation to $Re(s) \geq \frac{1}{2}$;

(iii) $(F \otimes G) \otimes (F \otimes G)$ is analytic for $Re(s) > 1$ and has a pole at $s = 1$.

Then, $(F \otimes G)(1 + it) \neq 0$ for all t.

Proof Let

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},\ G(s) = \sum_{n=1}^{\infty} \frac{a_G(n)}{n^s}$$

and suppose that $(F \otimes G)(1 + it_0) = 0$ for a real t_0. Let

$$f_1(n) = a_F(n)n^{-it_0},\ f_2(n) = \overline{a_F(n)}n^{it_0},\ g_1(n) = a_G(n),\ g_2(n) = \overline{a_G(n)},$$

and for $Re(s) > 1$, consider the following Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{|(f_1 * g_1)(n)|^2}{n^s} = \sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n)(f_2 * g_2)(n)}{n^s}.$$

Since f_1 and f_2 are completely multiplicative, by Lemma 3.19 we have

$$\begin{aligned} L(f_1f_2, s) &= \sum_{n=1}^{\infty} \frac{|a_F(n)|^2}{n^s} \\ &= L(F \otimes F, s) \\ &= (F \otimes F)(s). \end{aligned}$$

Similarly, we can derive the following

$$L(g_1g_2, s) = (G \otimes G)(s), \quad L(f_1g_2, s) = (F \otimes G)(s + it_0), \quad L(f_2g_1, s) = (G \otimes F)(s - it_0),$$

and

$$L(f_1f_2g_1g_2, 2s) = [(F \otimes G) \otimes (F \otimes G)] (2s).$$

So, by Lemma 3.21 and for $Re(s) > 1$, we have

$$f(s) = \frac{(F \otimes F)(s)(G \otimes G)(s)(F \otimes G)(s + it_0)(G \otimes F)(s - it_0)}{[(F \otimes G) \otimes (F \otimes G)] (2s)}.$$

Now by assumption of $(F \otimes G)(1 + it_0) = 0$ we have in fact the analyticity of $f(s)$ for $Re(s) > \frac{1}{2}$, and since the coefficients in the series are non-negative, by Lemma 3.13 the Dirichlet series representing $f(s)$ is convergent for $Re(s) > \frac{1}{2}$. So, for $\eta > 0$, we have

$$f\left(\frac{1}{2} + \eta\right) = \sum_{n=1}^{\infty} \frac{|(f_1 * g_1)(n)|^2}{n^{\frac{1}{2}+\eta}} \geq 1.$$

However, since $(F \otimes G) \otimes (F \otimes G)$ has a pole at $s = 1$,

$$[(F \otimes G) \otimes (F \otimes G)] \left(2\left(\frac{1}{2} + \eta\right)\right) = [(F \otimes G) \otimes (F \otimes G)] (1 + 2\eta) \to \infty$$

as $\eta \to 0^+$. This shows that

$$\lim_{\eta \to 0^+} f\left(\frac{1}{2} + \eta\right) = 0,$$

which is a contradiction.
This completes the proof. □

By choosing $G(s) = \zeta(s)$ in the previous theorem, we have

Corollary 3.23 *Let $F \in \bar{S}$ be analytic in $Re(s) \geq \frac{1}{2}$ and assume that $(F \otimes F)(s)$ is analytic in $Re(s) \geq \frac{1}{2}$, except for a simple pole at $s = 1$. If the coefficients of F are completely multiplicative, then $F(1 + it) \neq 0$, for all $t \in \mathbb{R}$.*

Note that the non-vanishing of $L_\chi(s)$ $(\chi \neq \chi_0)$ on the line $Re(s) = 1$, is a simple consequence of this corollary.

Chapter 4

Non-Vanishing of Symmetric Square L-Functions Inside the Critical Strip

4.1 Introduction

Studying the non-vanishing properties of L-functions inside the critical strip (i.e., $0 \leq Re(s) \leq 1$) is much more difficult than investigating the non-vanishing of L-functions at the edge of the critical strip (i.e., $Re(s) = 1$). The Generalized Riemann Hypothesis asserts that all L-functions should be non-vanishing on the strip $\frac{1}{2} < Re(s) \leq 1$. However, we are very far from a proof of this conjecture. Even in the case of the Riemann zeta-function, the current techniques of analytic number theory fail to prove the non-vanishing of the zeta-function on a narrow strip adjacent to the line $Re(s) = 1$. Nevertheless, there are results that establish the non-vanishing for infinite families of L-functions. To prove such results, one should study the asymptotic behavior of the values of L-functions on average.

In this chapter we prove a non-vanishing theorem for the symmetric square L-functions associated to newforms of weight 2 and prime level N. The main step in the proof of our result is establishing an upper bound for the mean values of the symmetric square L-functions in the critical strip.

4.2 An Upper Bound for the Mean Square

Let $\mathcal{F}_N$ be the set of newforms of weight 2 and prime level N and let $L(\mathrm{sym}^2 f, s)$ denote the symmetric square L-function associated to the newform f (see Section 2.3 and Subsection 1.2.4 for definitions). For a fixed point s_0 inside the critical strip, we derive an upper bound for the following mean square of symmetric square L-functions

$$\sum_{f \in \mathcal{F}_N} |L(\mathrm{sym}^2 f, s_0)|^2.$$

In [10], Iwaniec and Michel proved such an upper bound in the case of $Re(s_0) = \frac{1}{2}$. We closely follow their approach, and show that a similar result is true for a point inside the critical strip. The main result of this chapter is the following.

Theorem 4.1 *Let s_0 be a point in the strip $\frac{3}{4} \le \sigma_0 = Re(s_0) \le 1$. Then,*

$$\sum_{f \in \mathcal{F}_N} |L(\mathrm{sym}^2 f, s_0)|^2 \ll |s_0|^{9+\frac{6}{\epsilon}} N^{1+\epsilon}$$

for any $\epsilon > 0$. The implied constant depends only on ϵ.

To prove this theorem, we need several lemmas. The next section is devoted to the proof of these necessary lemmas.

4.3 Lemmas

We start by finding a representation for $L(\mathrm{sym}^2 f, s_0)$ as a sum of two absolutely convergent series. Recall that $L_\infty(\mathrm{sym}^2, s)$ is the product of gamma-factors in the functional equation of the symmetric square L-functions (see Section 2.3).

Lemma 4.2 *Let $A > 2$ be an integer and let $G(s) = \cos\left(\frac{\pi s}{4A}\right)^{-3A}$. For any s_0 with $0 \le Re(s_0) \le 1$, we have*

$$L(\mathrm{sym}^2 f, s_0) = \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) + \varepsilon(s_0) \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{1-s_0}} V_{1-s_0}\left(\frac{n}{N}\right)$$

where

$$V_{s_0}(y) = \int_{(2)} G(s) \frac{L_\infty(\mathrm{sym}^2, s_0 + s)}{L_\infty(\mathrm{sym}^2, s_0)} \zeta_N(2s_0 + 2s) y^{-s} \frac{ds}{s}$$

and $\varepsilon(s_0) = N^{1-2s_0} L_\infty(\mathrm{sym}^2, 1 - s_0)/L_\infty(\mathrm{sym}^2, s_0)$. *Here,*

$$\int_{(c)} g(s) ds := \lim_{T\to+\infty} \int_{-T}^{T} g(c + it) i dt.$$

Proof By starting with the first sum in the right-hand side of the statement and using the definition of $V_{s_0}(y)$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) &= \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{s_0}} \int_{(2)} G(s) \frac{L_\infty(\mathrm{sym}^2, s_0 + s)}{L_\infty(\mathrm{sym}^2, s_0)} \zeta_N(2s_0 + 2s) \left(\frac{N}{n}\right)^s \frac{ds}{s} \\
&= \int_{(2)} G(s) \frac{L_\infty(\mathrm{sym}^2, s_0 + s)}{L_\infty(\mathrm{sym}^2, s_0)} \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{s_0+s}} \zeta_N(2s_0 + 2s) N^s \frac{ds}{s}.
\end{aligned}$$

Now by using the definitions of $L(\mathrm{sym}^2 f, s_0)$ and $\Lambda(\mathrm{sym}^2 f, s_0)$ (see Section 2.3); and using the functional equation (2.7), we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) &= \int_{(2)} G(s) \frac{N^{s_0+s} L_\infty(\mathrm{sym}^2, s_0 + s) L(\mathrm{sym}^2 f, s_0 + s)}{L_\infty(\mathrm{sym}^2, s_0)} \frac{1}{N^{s_0}} \frac{ds}{s} \\
&= \int_{(2)} G(s) \frac{\Lambda(\mathrm{sym}^2 f, s_0 + s)}{L_\infty(\mathrm{sym}^2, s_0)} \frac{1}{N^{s_0}} \frac{ds}{s} \\
&= \int_{(2)} G(s) \frac{\Lambda(\mathrm{sym}^2 f, 1 - s_0 - s)}{L_\infty(\mathrm{sym}^2, s_0)} \frac{1}{N^{s_0}} \frac{ds}{s}.
\end{aligned}$$

Moving the line of integration to (-2) yields

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) &= G(0) \frac{\Lambda(\mathrm{sym}^2 f, 1 - s_0)}{L_\infty(\mathrm{sym}^2, s_0)} \frac{1}{N^{s_0}} \\
&\quad + \int_{(-2)} G(s) \frac{\Lambda(\mathrm{sym}^2 f, 1 - s_0 - s)}{L_\infty(\mathrm{sym}^2, s_0)} \frac{1}{N^{s_0}} \frac{ds}{s} \\
&= L(\mathrm{sym}^2 f, s_0) + \int_{(-2)} G(s) \frac{\Lambda(\mathrm{sym}^2 f, 1 - s_0 - s)}{L_\infty(\mathrm{sym}^2, s_0)} \frac{1}{N^{s_0}} \frac{ds}{s}.
\end{aligned}$$

Now by applying the change of variable $s \mapsto -u$, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) &= L(\mathrm{sym}^2 f, s_0) - \int_{(2)} G(-u) \frac{\Lambda(\mathrm{sym}^2 f, 1-s_0+u)}{L_\infty(\mathrm{sym}^2, s_0)} \frac{1}{N^{s_0}} \frac{-du}{-u} \\ &= L(\mathrm{sym}^2 f, s_0) \\ &- \int_{(2)} G(s) \frac{L_\infty(\mathrm{sym}^2, 1-s_0+s) L(\mathrm{sym}^2 f, 1-s_0+s)}{L_\infty(\mathrm{sym}^2, s_0)} N^{1-2s_0+s} \frac{ds}{s}.\end{aligned}$$

By changing the order of addition and integration in the above equality, the second summand of the right-hand side becomes

$$N^{1-2s_0} \frac{L_\infty(\mathrm{sym}^2, 1-s_0)}{L_\infty(\mathrm{sym}^2, s_0)} \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{1-s_0}}$$

$$\times \int_{(2)} G(s) \frac{L_\infty(\mathrm{sym}^2, 1-s_0+s)}{L_\infty(\mathrm{sym}^2, 1-s_0)} \zeta_N(2-2s_0+2s) \left(\frac{N}{n}\right)^s \frac{ds}{s},$$

which is equal to

$$\varepsilon(s_0) \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{1-s_0}} V_{1-s_0}\left(\frac{n}{N}\right).$$

This completes the proof. □

In the sequel we need the following fact about the gamma-function, known as *Stirling's formula*. In any vertical strip $|\sigma| \le a$,

$$|\Gamma(\sigma+it)| = \sqrt{2\pi} e^{-\frac{1}{2}\pi|t|} |t|^{\sigma-\frac{1}{2}} (1 + r(\sigma, t))$$

where $r(\sigma, t) \to 0$ as $|t| \to \infty$ (see [15], Exercise 6.3.15).

In the next lemma we study the growth of $V_{s_0}(y)$.

Lemma 4.3 *For any $y > 0$, and any $s_0 = \sigma_0 + it_0$ with $\frac{3}{4} \le \sigma_0 \le 1$, we have*

(i) $V_{s_0}(y) \ll \mathbf{d}(N) |s_0|^{\frac{3}{2}A} y^{-A}$.

(ii) $V_{s_0}(y) \ll \mathbf{d}(N) \left(\max\{y^{\frac{1}{2}},\ y^{\frac{1}{4}}\} + 1\right)$.

(iii) $V_{s_0}(y) \ll \mathbf{d}(N) \left(1 + \frac{y}{|s_0|^{\frac{3}{2}}}\right)^{-A} \log\left(2 + \frac{1}{y}\right)$.

(iv) For $j \geq 0$, let $V_{s_0}^{(j)}(y)$ denote the j-th derivative of $V_{s_0}(y)$ with respect to y. Then,

$$V_{s_0}^{(j)}(y) \ll \mathbf{d}(N) y^{-j} \left(1 + \frac{y}{|s_0|^{\frac{3}{2}}}\right)^{-A} \log\left(2 + \frac{1}{y}\right).$$

Here, $\mathbf{d}(N)$ stands for the number of divisors of N.

Proof *(i)* By shifting the line of integration to (A), and by using the definition of the integral, we have

$$V_{s_0}(y) = \int_{(A)} G(s) \frac{L_\infty(\mathrm{sym}^2, s + s_0)}{L_\infty(\mathrm{sym}^2, s_0)} \zeta_N(2s + 2s_0) y^{-s} \frac{ds}{s}$$

$$= \int_{-\infty}^{+\infty} G(A + it) \frac{L_\infty(\mathrm{sym}^2, A + it + \sigma_0 + it_0)}{L_\infty(\mathrm{sym}^2, \sigma_0 + it_0)} \zeta_N(2A + 2it + 2\sigma_0 + 2it_0) y^{-A-it} \frac{idt}{A + it}. \tag{4.1}$$

For the zeta-factor $\zeta_N(s)$ in (4.1), we have the following estimation

$$\begin{aligned} |\zeta_N(2(A+\sigma_0) + 2i(t+t_0))| &= \left| \frac{\zeta(2A + 2\sigma_0 + 2it + 2it_0)}{\prod_{p|N} \left(1 - p^{-(2A+2\sigma_0+2it+2it_0)}\right)^{-1}} \right| \\ &\leq \zeta(2A) \prod_{p|N} \left|1 - p^{-(2A+2\sigma_0+2it+2it_0)}\right| \\ &\ll \prod_{p|N} (1 + p^{-(2A+2\sigma_0)}) \\ &\leq \prod_{p|N} (1 + 1) \\ &\leq \mathbf{d}(N). \end{aligned}$$

By applying Stirling's formula for L_∞-factor in (4.1), we have

$$\left| \frac{L_\infty(\mathrm{sym}^2, (A + \sigma_0) + i(t + t_0))}{L_\infty(\mathrm{sym}^2, \sigma_0 + it_0)} \right|$$

$$= \left| \frac{\pi^{-\frac{3}{2}(A+\sigma_0+it+it_0)} \Gamma\left(\frac{A+\sigma_0+it+it_0+1}{2}\right)^2 \Gamma\left(\frac{A+\sigma_0+it+it_0+2}{2}\right)}{\pi^{-\frac{3}{2}(\sigma_0+it_0)} \Gamma\left(\frac{\sigma_0+it_0+1}{2}\right)^2 \Gamma\left(\frac{\sigma_0+it_0+2}{2}\right)} \right|$$

$$
\begin{aligned}
&\leq \pi^{-\frac{3}{2}A}\frac{\left(e^{-\frac{1}{2}\pi|\frac{t+t_0}{2}|}|\frac{t+t_0}{2}|^{\frac{A+\sigma_0}{2}}\right)^2 e^{-\frac{1}{2}\pi|\frac{t+t_0}{2}|}|\frac{t+t_0}{2}|^{\frac{A+\sigma_0+1}{2}}}{\left(e^{-\frac{1}{2}\pi|\frac{t_0}{2}|}|\frac{t_0}{2}|^{\frac{\sigma_0}{2}}\right)^2 e^{-\frac{1}{2}\pi|\frac{t_0}{2}|}|\frac{t_0}{2}|^{\frac{\sigma_0+1}{2}}} \\
&\ll e^{\frac{3\pi}{4}(|t_0|-|t+t_0|)}\left|\frac{t+t_0}{t_0}\right|^{\frac{3\sigma_0+1}{2}}\left|\frac{t+t_0}{2}\right|^{\frac{3A}{2}} \\
&= \left(e^{\frac{3\pi}{4}|t|(|\frac{t_0}{t}|-|1+\frac{t_0}{t}|)}\left|\frac{t+t_0}{t_0}\right|^{\frac{3\sigma_0+1}{2}}\left|\frac{t+t_0}{2t_0}\right|^{\frac{3A}{2}}\right)|t_0|^{\frac{3A}{2}} \\
&\ll |s_0|^{\frac{3A}{2}} g(t,s_0)
\end{aligned}
$$

where $g(t,s_0)$ has exponential decay when $|t|\to\infty$.

Applying the above estimates in (4.1) yields

$$
\begin{aligned}
V_{s_0}(y) &\ll y^{-A}\int_{-\infty}^{+\infty}|G(A+it)||s_0|^{\frac{3A}{2}}g(t,s_0)\mathbf{d}(N)\frac{dt}{\sqrt{A^2+t^2}} \\
&\ll \mathbf{d}(N)|s_0|^{\frac{3A}{2}}y^{-A}.
\end{aligned}
$$

This proves part *(i)*.

(ii) Shifting the line of integration to $(-\frac{1}{2})$, and calculating the residues at $s=0$ and $s=\frac{1}{2}-s_0$ yield

$$
\begin{aligned}
V_{s_0}(y) &= \zeta_N(2s_0)+G\left(\frac{1}{2}-s_0\right)\frac{L_\infty(\mathrm{sym}^2,\frac{1}{2})}{L_\infty(\mathrm{sym}^2,s_0)}\frac{y^{s_0-\frac{1}{2}}}{\frac{1}{2}-s_0}\prod_{p|N}\left(1-\frac{1}{p}\right) \\
&+ \int_{(-\frac{1}{2})}G(s)\frac{L_\infty(\mathrm{sym}^2,s+s_0)}{L_\infty(\mathrm{sym}^2,s_0)}\zeta_N(2s+2s_0)y^{-s}\frac{ds}{s}. \qquad (4.2)
\end{aligned}
$$

We show that each summand of the above equality is bounded by a constant multiple of

$$
\mathbf{d}(N)\left(\max\{y^{\frac{1}{2}},\ y^{\frac{1}{4}}\}+1\right).
$$

For the first summand in (4.2), since $\frac{3}{4} \le \sigma_0 \le 1$, we have

$$\begin{aligned}
|\zeta_N(2s_0)| &= \frac{|\zeta(2s_0)|}{\left|\prod_{p|N}(1-p^{-2s_0})^{-1}\right|} \\
&\le \zeta(2\sigma_0)\prod_{p|N}\left|(1-p^{-2s_0})\right| \\
&\le \zeta\left(\frac{3}{2}\right)\prod_{p|N}(1+p^{-2\sigma_0}) \\
&\ll \mathbf{d}(N) \\
&\le \mathbf{d}(N)\left(\max\{y^{\frac{1}{2}},\ y^{\frac{1}{4}}\}+1\right).
\end{aligned}$$

For the second summand in (4.2), we observe the following:

$$L_\infty\left(\mathrm{sym}^2,\frac{1}{2}\right) \ll 1; \quad \frac{1}{\frac{1}{2}-s_0} \ll 1; \quad \prod_{p|N}\left(1-\frac{1}{p}\right) \ll \mathbf{d}(N);$$

and

$$\left|y^{s_0-\frac{1}{2}}\right| = y^{\sigma_0-\frac{1}{2}} \ll \max\{y^{\frac{1}{2}},\ y^{\frac{1}{4}}\}+1.$$

Now using the definition of $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$, for large t_0, we have

$$\begin{aligned}
\left|\frac{G(\frac{1}{2}-s_0)}{L_\infty(\mathrm{sym}^2,s_0)}\right| &\ll \frac{|e^{\frac{i\pi}{4A}(\sigma_0+it_0)}+e^{-\frac{i\pi}{4A}(\sigma_0+it_0)}|^{-3A}}{e^{-\frac{3\pi}{4}|t_0|}|\frac{t_0}{2}|^{\frac{3\sigma_0+1}{2}}} \\
&\ll \frac{|e^{(\frac{i\pi\sigma_0}{4A}-\frac{\pi t_0}{4A})}+e^{(-\frac{i\pi\sigma_0}{4A}+\frac{\pi t_0}{4A})}|^{-3A}}{e^{-\frac{3\pi}{4}|t_0|}} \\
&\ll \frac{|e^{\frac{\pi t_0}{4A}}(e^{\frac{i\pi\sigma_0-2\pi t_0}{4A}}+e^{-\frac{i\pi\sigma_0}{4A}})|^{-3A}}{e^{-\frac{3\pi}{4}|t_0|}} \\
&\ll 1.
\end{aligned}$$

Putting these together, we have

$$\begin{aligned}
&G\left(\frac{1}{2}-s_0\right)\frac{L_\infty(\mathrm{sym}^2,\frac{1}{2})}{L_\infty(\mathrm{sym}^2,s_0)}\frac{y^{s_0-\frac{1}{2}}}{\frac{1}{2}-s_0}\prod_{p|N}\left(1-\frac{1}{p}\right) \\
&\qquad \ll \mathbf{d}(N)\left(\max\{y^{\frac{1}{2}},\ y^{\frac{1}{4}}\}+1\right).
\end{aligned}$$

The estimation of the integral

$$\int_{(-\frac{1}{2})} G(s)\frac{L_\infty(\mathrm{sym}^2, s+s_0)}{L_\infty(\mathrm{sym}^2, s_0)}\zeta_N(2s+2s_0)y^{-s}\frac{ds}{s}$$

$$= \int_{-\infty}^{+\infty} G(-\frac{1}{2}+it)\frac{L_\infty(\mathrm{sym}^2, -\frac{1}{2}+it+\sigma_0+it_0)}{L_\infty(\mathrm{sym}^2, \sigma_0+it_0)}\zeta_N(-1+2it+2\sigma_0+2it_0)y^{\frac{1}{2}-it}\frac{idt}{-\frac{1}{2}+it}$$

in (4.2), is very similar to the one which we had in part *(i)*. The main difference is that this time we have to estimate the zeta-function inside the critical strip. By the classical inequality

$$\zeta(s) = O(t^{\frac{1}{2}})$$

for $\sigma = Re(s) \geq \frac{1}{2}$ and large t (see [15], Exercise 4.2.4), since $\dfrac{1}{2} \leq 2\sigma_0 - 1 \leq 1$, we obtain

$$\begin{aligned}\int_{(-\frac{1}{2})} G(s)\frac{L_\infty(\mathrm{sym}^2, s+s_0)}{L_\infty(\mathrm{sym}^2, s_0)}\zeta_N(2s+2s_0)y^{-s}\frac{ds}{s} &\ll \mathbf{d}(N)y^{\frac{1}{2}} \\ &\leq \mathbf{d}(N)\left(\max\{y^{\frac{1}{2}},\ y^{\frac{1}{4}}\}+1\right).\end{aligned}$$

This completes the proof of part *(ii)*.

(iii) If y is large, $\log(2+\frac{1}{y})$ is bounded (for example, by $\log 3$). Also

$$\left(\frac{y}{|s_0|^{\frac{3}{2}}}\right)^{-A} \sim \left(1+\frac{y}{|s_0|^{\frac{3}{2}}}\right)^{-A}.$$

So, we can replace the upper bound in *(i)* by *(iii)*. If y is near to 0, the upper bound is justified since $(1+\dfrac{y}{|s_s|^{\frac{3}{2}}})^{-A}$ is bounded by 1, and $\max\{y^{\frac{1}{2}},\ y^{\frac{1}{3}}\}+1$ is bounded by $\log(2+\frac{1}{y})$. This completes the proof of part *(iii)*.

(iv) Let

$$g_N(s, s_0) = G(s)\frac{L_\infty(\mathrm{sym}^2, s+s_0)}{L_\infty(\mathrm{sym}^2, s_0)}\zeta_N(2s+2s_0)\frac{1}{s}.$$

We have

$$V_{s_0}(y) = \int_{(2)} g_N(s, s_0)y^{-s}ds.$$

Now we prove, by induction, that for any $n \geq 0$,

$$V_{s_0}^{(n)}(y) = (-1)^n y^{-n} \int_{(2)} P_n(s) g_N(s, s_0) y^{-s} ds \tag{4.3}$$

where $P_n(s)$ is a polynomial of degree n, and for any n,

$$P_{n+1}(s) = (s+n)P_n(s).$$

For $n = 0$ the statement is true with $P_0(s) = 1$. Assuming (4.3) and taking derivatives, we get

$$\begin{aligned}
V_{s_0}^{(n+1)}(y) &= -n(-1)^n y^{-n-1} \int_{(2)} P_n(s) g_N(s, s_0) y^{-s} ds \\
&- (-1)^n y^{-n} \int_{(2)} s P_n(s) g_N(s, s_0) y^{-s-1} ds \\
&= (-1)^{n+1} y^{-n-1} \int_{(2)} (s+n) P_n(s) g_N(s, s_0) y^{-s} ds \\
&= (-1)^{n+1} y^{-(n+1)} \int_{(2)} P_{n+1}(s) g_N(s, s_0) y^{-s} ds.
\end{aligned}$$

Now similar to the proof of the previous parts, one can show that

$$V_{s_0}^{(j)}(y) \ll \mathbf{d}(N) y^{-j} (1 + \frac{y}{|s_0|^{\frac{3}{2}}})^{-A} \log(2 + \frac{1}{y}).$$

The implied constant depends upon N, s_0 and j.

The proof of the lemma is complete. $\square$

In the proof of our main theorem we need a smooth partition of unity. The next lemma will guarantee the existence of such a partition.

Lemma 4.4 *There exists a non-negative C^∞ function h with support $[1, 2]$, such that for any $x > 0$,*

$$\sum_{k=-\infty}^{\infty} h(\frac{x}{2^{k/2}}) = 1.$$

Proof We know that there is an absolutely increasing and C^∞ function $h : [1,\ \sqrt{2}] \to [0,\ 1]$ such that

$$h(1) = 0,\ h(\sqrt{2}) = 1,\ h'(1) = h'(\sqrt{2}) = 0.$$

We extend the domain of h to $[\sqrt{2},\ 2]$ by setting

$$h(x) = 1 - h(\frac{x}{\sqrt{2}})$$

where $x \in [\sqrt{2},\ 2]$. For $x \in \mathbb{R} \setminus [1,\ 2]$ we put $h(x) = 0$. It is apparent that h is non-negative, smooth on $\mathbb{R}$, and that for any $1 \le t \le \sqrt{2}$,

$$h(t) + h(\sqrt{2}t) = 1. \tag{4.4}$$

We claim that h satisfies the desired identity. To do this, first notice that for any $x > 0$, there is a unique $n \in \mathbb{Z}$ such that $2^{\frac{n}{2}} \le x < 2^{\frac{(n+1)}{2}}$. Note that

$$h\left(\frac{x}{2^{k/2}}\right) \neq 0 \iff 2^{\frac{k}{2}} < x < 2^{\frac{k+2}{2}} \iff k = n-1,\ n.$$

Therefore, by (4.4), we have

$$\sum_{k=-\infty}^{\infty} h(\frac{x}{2^{k/2}}) = h(\frac{x}{2^{(n-1)/2}}) + h(\frac{x}{2^{n/2}}) = 1.$$

The proof is complete. □

In the proof of Theorem 4.1, we need some facts from the theory of modular forms. We start by the following definition.

Definition 4.5 *For a cusp form $f \in S_k(N)$ of weight k and level N, let*

$$\omega_f = \frac{(4\pi)^{k-1}}{(k-2)!}\langle f, f\rangle$$

where $\langle f, f\rangle$ stands for the Petersson inner product.

The next two lemmas give estimates for the values of ω_f.

Lemma 4.6 *For a newform $f \in S_2(N)$, we have*

$$\omega_f \ll N(\log N)^3.$$

Proof By (2.6) and for $k = 2$, we have

$$\omega_f = \frac{N}{2\pi^2} L(\mathrm{sym}^2 f, 1).$$

By applying the Phragmén-Lindelöf theorem for $L(\mathrm{sym}^2 f, s)$, and for $Re(s) = 1$ (see [13], p. 336),

$$L(\mathrm{sym}^2 f, 1) \ll (\log N)^3 .$$

This completes the proof. □

Lemma 4.7 *We have*

$$\sum_{f \in \mathcal{F}_N} \omega_f^{-1} = 1 + O(N^{-\frac{3}{2}}).$$

Proof Let $a_f(n)$ denote the n-th normalized Fourier coefficient of f. We have,

$$\sum_{f \in \mathcal{F}_N} \frac{a_f(m) a_f(n)}{4\pi \langle f, f \rangle} = \delta_{mn} + O\left(N^{-\frac{3}{2}} (\text{g.c.d.}(m,n))^{\frac{1}{2}} \sqrt{mn}\right)$$

(see [14], Proposition 1). Here

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

is the *Kronecker symbol*. Putting $m = n = 1$ in the above formula implies the result. The proof is complete. □

4.4 The Proof

To start the proof of our main theorem, we need one more ingredient. The proof of the following theorem is very long and technical. So, we only state it without proof.

Theorem 4.8 *Let s_0 be a point inside the critical strip. Also let g be a smooth function with support* $[1,\ 2]$ *satisfying*

$$g^{(j)}(x) \ll |s_0|^j$$

for any $j \geq 0$. For $X \geq 1$ we define the partial sums

$$S_f(X) = \sum_n a_f(n^2) g(\frac{n}{X})$$

and their mean square

$$S(X) = \sum_{f \in \mathcal{F}_N} \omega_f^{-1} |S_f(X)|^2.$$

Then we have

$$S(X) \ll |s_0|^{3+\epsilon} (NX)^\epsilon (N^{-1} X^2 + X)$$

for any $\epsilon > 0$. *The implied constant depends only on* ϵ.

Proof See [10], Theorem 5.1. □

We are ready to prove the main theorem of this chapter.

Proof of Theorem 4.1 Let ϵ be the reciprocal of a natural number bigger than 2 and let $A = 3 + \frac{2}{\epsilon}$. It is plain that

$$A \in \mathbb{N}, \ 0 < \epsilon < \frac{1}{2}, \ \frac{A+\epsilon}{A-2} = 1 + \epsilon.$$

Now write $L(\mathrm{sym}^2 f, s_0) = L_1(f, s_0) + L_2(f, s_0) + L_3(f, s_0) + L_4(f, s_0)$, where

$$L_1(f, s_0) = \sum_{n \le N^{1+\epsilon}} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right),$$

$$L_2(f, s_0) = \sum_{n \ge N^{1+\epsilon}} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right),$$

$$L_3(f, s_0) = N^{1-2s_0} \frac{L_\infty(\mathrm{sym}^2, 1-s_0)}{L_\infty(\mathrm{sym}^2, s_0)} \sum_{n \le N^{1+\epsilon}} \frac{a_f(n^2)}{n^{1-s_0}} V_{1-s_0}\left(\frac{n}{N}\right),$$

$$L_4(f, s_0) = N^{1-2s_0} \frac{L_\infty(\mathrm{sym}^2, 1-s_0)}{L_\infty(\mathrm{sym}^2, s_0)} \sum_{n \ge N^{1+\epsilon}} \frac{a_f(n^2)}{n^{1-s_0}} V_{1-s_0}\left(\frac{n}{N}\right).$$

Our first goal is to estimate the $L_i(f, s_0)$'s. We start with $L_2(f, s_0)$. By Deligne's bound (see Theorem 1.4), and part *(iii)* of Lemma 4.3,

$$\begin{aligned}
|L_2(f, s_0)| &= \left| \sum_{n \ge N^{1+\epsilon}} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) \right| \\
&\le \sum_{n \ge N^{1+\epsilon}} \frac{|a_f(n^2)|}{n^{\sigma_0}} \left| V_{s_0}\left(\frac{n}{N}\right) \right| \\
&\ll \sum_{n \ge N^{1+\epsilon}} \frac{\mathbf{d}(n^2)}{n^{\sigma_0}} \mathbf{d}(N) \frac{|s_0|^{\frac{3}{2}A}}{(|s_0|^{\frac{3}{2}} + \frac{n}{N})^A} \log\left(2 + \frac{N}{n}\right).
\end{aligned}$$

Since $\dfrac{A+\epsilon}{A-2} = 1+\epsilon$, we have

$$n \geq N^{1+\epsilon} \iff \frac{n}{N} \geq \left(n^2 N^{\epsilon}\right)^{\frac{1}{A}}.$$

By using the classical inequality

$$\mathbf{d}(n) \ll n^{\delta}$$

for any $\delta > 0$ (see [15], Exercise 1.3.2), and by ignoring $|s_0|^{\frac{3}{2}}$ in the denominator, we get

$$\begin{aligned}
L_2(f,s_0) &\ll \mathbf{d}(N)|s_0|^{\frac{3}{2}A} \sum_{n\geq N^{1+\epsilon}} \frac{n^{\epsilon}}{n^{\sigma_0}} \frac{1}{(\frac{n}{N})^A} \\
&\ll \mathbf{d}(N)|s_0|^{\frac{3}{2}A} \sum_{n\geq N^{1+\epsilon}} \frac{1}{n^{\sigma_0-\epsilon}} \frac{1}{n^2 N^{\epsilon}} \\
&= \frac{\mathbf{d}(N)|s_0|^{\frac{3}{2}A}}{N^{\epsilon}} \sum_{n\geq N^{1+\epsilon}} \frac{1}{n^{2+\sigma_0-\epsilon}} \\
&\ll \frac{\mathbf{d}(N)|s_0|^{\frac{3}{2}A}}{N^{\epsilon}}. \qquad (4.5)
\end{aligned}$$

With a similar argument we attain

$$\begin{aligned}
L_4(f,s_0) &\ll \frac{\mathbf{d}(N)N^{1-2\sigma_0}}{N^{\epsilon}} \left|\frac{L_\infty(\mathrm{sym}^2, 1-s_0)}{L_\infty(\mathrm{sym}^2, s_0)}\right| |1-s_0|^{\frac{3}{2}A} \\
&\ll \frac{\mathbf{d}(N)N^{1-2\sigma_0}}{N^{\epsilon}} |1-s_0|^{\frac{3}{2}A}. \qquad (4.6)
\end{aligned}$$

This is true, since by Stirling's formula, the ratio of the L_∞-factors is bounded.

To estimate $L_1(f,s_0)$, we first rewrite it as a new sum involving the function h in Lemma 4.4. For simplicity, we use X for $2^{k/2}$. Using the main identity for h in Lemma 4.4, we have

$$\begin{aligned}
L_1(f,s_0) &= \sum_{n\leq N^{1+\epsilon}} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) \\
&= \sum_{n\leq N^{1+\epsilon}} \left(\frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) \sum_{k=-\infty}^{\infty} h\left(\frac{n}{X}\right)\right) \\
&= \sum_{n\leq N^{1+\epsilon}} \sum_{k} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) h\left(\frac{n}{X}\right).
\end{aligned}$$

In the last equality, since one sum is finite, we can interchange the order of the addition. Note that the support of h is $[1, 2]$, so, we can assume that $X < n < 2X$. Also, since $n \geq 1$, k is in fact ≥ -1. Therefore,

$$\begin{aligned}
L_1(f, s_0) &= \sum_{k\geq -1} \sum_{X<n<X} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) h\left(\frac{n}{X}\right) \\
&= \sum_{k\geq -1} \left(\frac{\mathbf{d}(N)\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^{-A} \log\left(2+\frac{N}{X}\right)}{\mathbf{d}(N)\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^{-A} \log\left(2+\frac{N}{X}\right)} \sum_{X<n<2X} \frac{a_(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) h\left(\frac{n}{X}\right) \right) \\
&= \sum_{k\geq -1} \left(\frac{\mathbf{d}(N)\log\left(2+\frac{N}{X}\right)}{X^{s_0}\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^{A}} \sum_{X<n<2X} a_f(n^2) g\left(\frac{n}{X}\right) \right),
\end{aligned}$$

where

$$g(x) = \frac{1}{\mathbf{d}(N)\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^{-A} \log\left(2+\frac{N}{X}\right)} x^{-s_0} V_{s_0}\left(\frac{X}{N}x\right) h(x).$$

To be consistent with the notations of Theorem 4.8, we put

$$S_f(X) = \sum a_f(n^2) g\left(\frac{n}{X}\right).$$

So,

$$L_1(f, s_0) = \sum_{k\geq -1} \frac{\mathbf{d}(N)\log\left(2+\frac{N}{X}\right)}{\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^{A}} \frac{S_f(X)}{X^{s_0}}.$$

In a similar fashion

$$L_3(f, s_0) = N^{1-2s_0} \frac{L_\infty(\mathrm{sym}^2, 1-s_0)}{L_\infty(\mathrm{sym}^2, s_0)} \sum_{k\geq -1} \frac{\mathbf{d}(N)\log\left(2+\frac{N}{X}\right)}{\left(1+\frac{X}{N|1-s_0|^{\frac{3}{2}}}\right)^{A}} \frac{S_f(X)}{X^{1-s_0}}.$$

By the Cauchy-Schwarz inequality

$$\left|L(\mathrm{sym}^2 f, s_0)\right|^2 = \left|\sum_{i=1}^{4} L_i(f, s_0)\right|^2 \leq 4 \sum_{i=1}^{4} \left|L_i(f, s_0)\right|^2.$$

So, we have

$$\sum_{f\in\mathcal{F}_N} \omega_f^{-1}|L(\mathrm{sym}^2 f, s_0)|^2$$

$$\ll \sum_f \omega_f^{-1}|L_1(f,s_0)|^2 + \sum_f \omega_f^{-1}|L_2(f,s_0)|^2 + \sum_f \omega_f^{-1}|L_3(f,s_0)|^2 + \sum_f \omega_f^{-1}|L_4(f,s_0)|^2.$$

Now we estimate the above four sums. For the first sum, by applying the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}
\sum_f \omega_f^{-1}\left|L_1(f,s_0)\right|^2 &= \sum_f \omega_f^{-1}\left|\sum_{k\geq -1} \frac{\mathbf{d}(N)\log\left(2+\frac{N}{X}\right)}{\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^A}\frac{S_f(X)}{X^{s_0}}\right|^2 \\
&\ll \sum_f \omega_f^{-1}\left(\sum_{k\geq -1}\frac{\mathbf{d}^2(N)\log^2\left(2+\frac{N}{X}\right)}{\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^{2A}}\sum_{k\geq -1}\frac{|S_f(X)|^2}{X^{2\sigma_0}}\right) \\
&\ll \mathbf{d}^2(N)\sum_{k\geq -1}\frac{\log^2(2+\frac{N}{X})}{(1+\frac{X}{N|s_0|^{\frac{3}{2}}})^{2A}}\sum_{k\geq -1}\frac{1}{X^{2\sigma_0}}\left(\sum_f \omega_f^{-1}|S_f(X)|^2\right).
\end{aligned} \tag{4.7}$$

It can be shown that the function $g(x)$ satisfies the conditions of Theorem 4.8. Moreover, note that the conditions $n \leq N^{1+\epsilon}$ and $X < n < 2X$ imply that

$$-1 \leq k < 2(1+\epsilon)\log_2 N.$$

So, by applying the result of Theorem 4.8 in (4.7), we deduce

$$\begin{aligned}
\sum_f \omega_f^{-1}\left|L_1(f,s_0)\right|^2 &\ll \mathbf{d}^2(N)N^\epsilon \sum_{k\geq -1}\frac{1}{X^{2\sigma_0}}|s_0|^{3+\epsilon}(NX)^\epsilon\left(N^{-1}X^2+X\right) \\
&\ll |s_0|^{3+\epsilon}N^\epsilon.
\end{aligned} \tag{4.8}$$

Here we are using the fact that $\frac{3}{4} \leq \sigma_0 \leq 1$. For convenience, whenever it is necessary, we replace a constant multiple of ϵ with ϵ.

Also note that (4.5) and Lemma 4.7 imply

$$\begin{aligned}\sum_f \omega_f^{-1}|L_2(f,s_0)|^2 &\ll \sum_f \omega_f^{-1}\left(\frac{\mathbf{d}(N)|s_0|^{\frac{3}{2}A}}{N^\epsilon}\right)^2 \\ &= \frac{\mathbf{d}^2(N)|s_0|^{3A}}{N^\epsilon}\sum_f \omega_f^{-1} \\ &\ll |s_0|^{3A}N^{-\epsilon} \\ &\leq |s_0|^{3A}N^\epsilon. \qquad (4.9)\end{aligned}$$

In a similar fashion we derive the following inequalities

$$\begin{aligned}\sum_f \omega_f^{-1}|L_3(f,s_0)|^2 &\ll |1-s_0|^{3+\epsilon}N^{2-4\sigma_0+\epsilon} \\ &\ll |s_0|^{3+\epsilon}N^\epsilon, \qquad (4.10)\end{aligned}$$

and

$$\sum_f \omega_f^{-1}|L_4(f,s_0)|^2 \ll |s_0|^{3A}N^\epsilon. \qquad (4.11)$$

Considering (4.8), (4.9), (4.10) and (4.11), we arrive at

$$\sum_{f\in\mathcal{F}_N} \omega_f^{-1}|L(\mathrm{sym}^2 f,s_0)|^2 \ll |s_0|^{3A}N^\epsilon.$$

Finally, by using the upper bound of Lemma 4.6, $\omega_f \ll N^{1+\epsilon}$, we conclude that

$$\begin{aligned}\sum_{f\in\mathcal{F}_N} |L(\mathrm{sym}^2 f,s_0)|^2 &= \sum_{f\in\mathcal{F}_N} \omega_f\omega_f^{-1}|L(\mathrm{sym}^2 f,s_0)|^2 \\ &\ll N^{1+\epsilon}\sum_{f\in\mathcal{F}_N} \omega_f^{-1}|L(\mathrm{sym}^2 f,s_0)|^2 \\ &\ll |s_0|^{3A}N^{1+\epsilon} = |s_0|^{9+\frac{6}{\epsilon}}N^{1+\epsilon}.\end{aligned}$$

The proof is now complete. □

4.5 A Non-Vanishing Result

In this final section, we show how the combination of the main result of this chapter (Theorem 4.1) with a known theorem about the values of the symmetric square L-functions on average will lead to a non-vanishing result. In [2], Akbary proved the following.

Theorem 4.9 *Let N be prime, then there exists $C > 0$ such that for any $s_0 = \sigma_0 + it_0$ with $1 - \frac{1}{46} < Re(s) \leq 1$, we have*

$$\sum_{f \in \mathcal{F}_N} L(\mathrm{sym}^2 f, s_0) = \zeta(1 + s_0)\zeta_N(2s_0)\frac{N-1}{12} + O\left(\frac{N^{\frac{91}{46}-\sigma_0}(\log N)^C}{|\Gamma(\frac{s_0+1}{2})|^2|\Gamma(\frac{s_0+2}{2})|}\right)$$

where the implied constant depends only on σ_0.

Proof See [2], Theorem 1 and formula (9). □

By using this theorem, we can prove the following.

Theorem 4.10 *Let N be a prime number and let $s_0 = \sigma_0 + it_0$ with $1 - \frac{1}{46} < \sigma_0 < 1$. Then for any $\epsilon > 0$, there are positive constants $C_{s_0,\epsilon}$ and $C'_{s_0,\epsilon}$ (depending only on s_0 and ϵ), such that for any prime $N > C'_{s_0,\epsilon}$, there exist at least $C_{s_0,\epsilon}N^{1-\epsilon}$ newforms f of weight 2 and level N for which $L(\mathrm{sym}^2 f, s_0) \neq 0$.*

Proof By the asymptotic formula of Theorem 4.9, and by Cauchy-Schwarz inequality we can write

$$\begin{aligned} N^2 &\ll \left|\sum_{f \in \mathcal{F}_N} L(\mathrm{sym}^2 f, s_0)\right|^2 \\ &\leq \#\left\{f \in \mathcal{F}_N : \ L(\mathrm{sym}^2 f, s_0) \neq 0\right\} \sum_{f \in \mathcal{F}_N} |L(\mathrm{sym}^2 f, s_0)|^2 \\ &\ll \#\left\{f \in \mathcal{F}_N : \ L(\mathrm{sym}^2 f, s_0) \neq 0\right\} |s_0|^{3A} N^{1+\epsilon}. \end{aligned}$$

Thus,

$$\#\left\{f \in \mathcal{F}_N : \ L(\mathrm{sym}^2 f, s_0) \neq 0\right\} \gg \frac{1}{|s_0|^{3A}} N^{1-\epsilon}.$$

The proof is complete. □

Finally we present a non-vanishing corollary of our theorem.

Corollary 4.11 *For any* $s_0 = \sigma_0 + it_0$ *with* $1 - \frac{1}{46} < \sigma_0 < 1$, *there are infinitely many symmetric square* L*-functions associated to newforms* f *such that* $L(\mathrm{sym}^2 f, s_0) \neq 0$.

Proof Note that there are infinitely many (large) primes. So, applying the previous theorem for newforms of different levels corresponding to different primes yields the result.
This completes the proof. $\square$

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